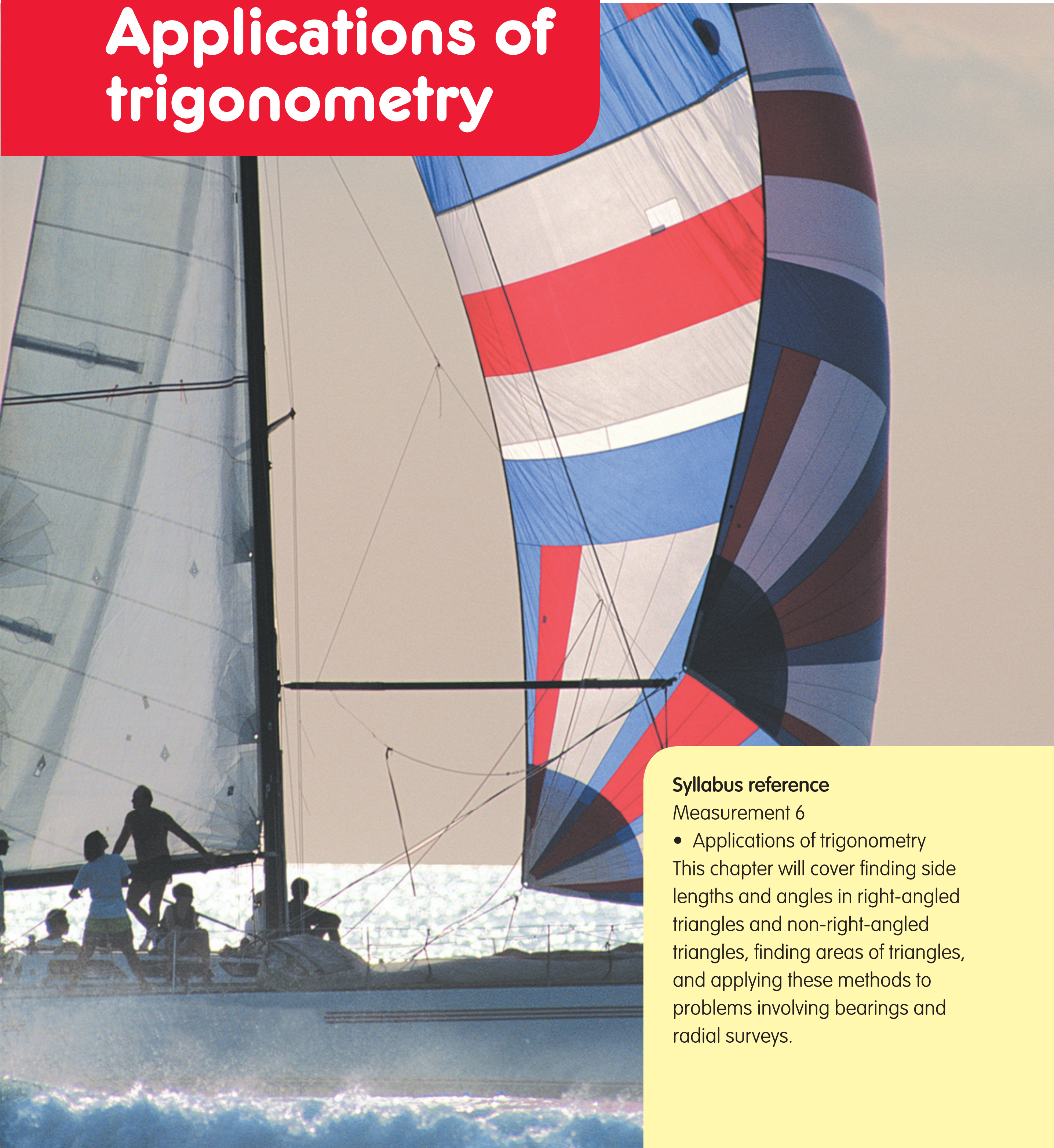
3

Applications of trigonometry



Syllabus reference

Measurement 6

- Applications of trigonometry

This chapter will cover finding side lengths and angles in right-angled triangles and non-right-angled triangles, finding areas of triangles, and applying these methods to problems involving bearings and radial surveys.

ARE YOU READY?

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Digital doc
SkillSHEET 3.1
doc-1320
Right-angled trigonometry — finding a side length

eBook*plus*

Digital doc
SkillSHEET 3.2
doc-1321
Using the inverse trigonometric ratios

eBook*plus*

Digital doc
SkillSHEET 3.4
doc-1323
Right-angled trigonometry — finding an angle

eBook*plus*

Digital doc
SkillSHEET 3.5
doc-1324
Converting nautical miles to kilometres

eBook*plus*

Digital doc
SkillSHEET 3.6
doc-1325
Angle sum of a triangle

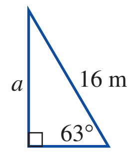
eBook*plus*

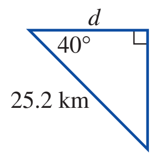
Digital doc
SkillSHEET 3.7
doc-1326
Solving fractional equations

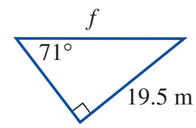
Try the questions below. If you have difficulty with any of them, extra help can be obtained by completing the matching SkillSHEET. Either click on the SkillSHEET icon next to the question on the *Maths Quest HSC Course* eBookPLUS icon.

Right-angled trigonometry — finding a side length

1 In each of the following find the length of the side marked with the pronumerals correct to 2 decimal places.

a


b


c


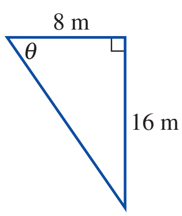
Using the inverse trigonometric ratios

2 Find angle θ, where θ is acute, correct to the nearest degree.

a $\sin\theta = 0.7$ **b** $\tan\theta = 1.5$ **c** $\cos\theta = 0.8$

Right-angled trigonometry — finding an angle

3 In each of the following find the size of the angle marked with the pronumerals correct to the nearest degree.

a


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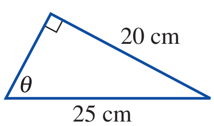

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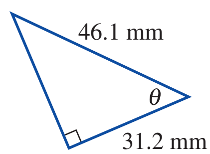


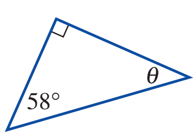
Converting nautical miles to kilometres

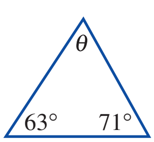
4 Use 1 nautical mile = 1.852 km to convert:

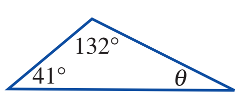
a 4 nautical miles to kilometres
b 50 kilometres to nautical miles
c 1.2 nautical miles to metres
d 3560 metres to nautical miles.

Angle sum of a triangle

5 Find the angle marked with the pronumeral in each of the following.

a


b


c


Solving fractional equations

6 Solve each of the following equations, where appropriate give your answer correct to 2 decimal places.

a $\frac{x}{5} = 3$ **b** $\frac{x}{4} = \frac{3}{8}$ **c** $\frac{x}{3.6} = \frac{9.5}{2.4}$ **d** $\frac{9}{x} = \frac{2}{5}$

3A Review of right-angled triangles

Previously we have studied right-angled triangles and discovered that we can calculate a side length of a triangle when given the length of one other side and one of the acute angles.

To do this we need to use one of the formulas for the three trigonometric ratios.

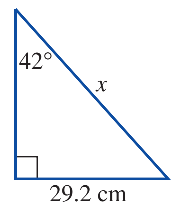
$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

WORKED EXAMPLE 1

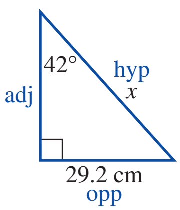
Find the length of the side marked *x* in the figure on the right (correct to 1 decimal place).



THINK | **WRITE**

Method 1: Technology-free

1 Label the sides of the diagram.



2 Choose the sine ratio and write the formula.

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

3 Substitute for the opposite side and hypotenuse.

$$\sin 42° = \frac{29.2}{x}$$

4 Make x the subject of the formula.

$$x\sin 42° = 29.2$$

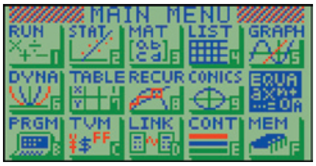
$$x = \frac{29.2}{\sin 42°}$$

5 Calculate the value of x.

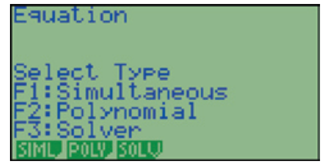
$$x = 43.6 \text{ m}$$

Method 2: Technology-enabled

1 From the **MENU** select **EQUA**.

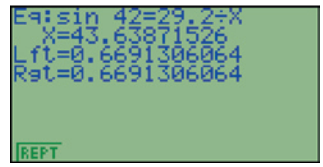


2 Press F3 (**SOLV**).



3 Delete any equation, enter the equation **sin 42 = 29.2 ÷ X** and press EXE.
Note: Your calculator may display a different value of **X** at this stage. This is just the last value of **X** stored in the calculator's memory.

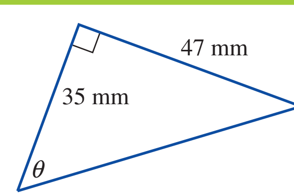
4 Press F6 (**SOLV**) to solve the equation.



The same formulas can be used to calculate the size of an angle if we are given two side lengths in the triangle.

WORKED EXAMPLE 2

Calculate the size of the angle marked θ in the figure on the right (correct to the nearest degree).



THINK | **WRITE**

Method 1: Technology-free

1. Label the sides of the triangle. — Opposite = 47 mm; Adjacent = 35 mm

2. Choose the tangent ratio and write the formula. — $\tan\,\theta = \dfrac{\text{opposite side}}{\text{adjacent side}}$

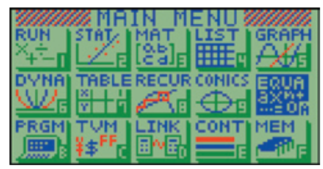
3. Substitute for the opposite side and the adjacent side. — $\tan\,\theta = \dfrac{47}{35}$

4. Make θ the subject of the formula. — $\theta = \tan^{-1}\left(\dfrac{47}{35}\right)$

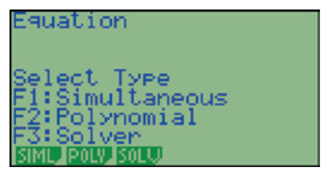
5. Calculate θ. — $\theta = 53°$

Method 2: Technology-enabled

1. From the **MENU** select **EQUA**.

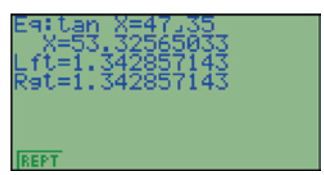


2. Press F3 (**SOLV**).



3. Delete any existing equation, then enter the equation **tan X = 47** $a\frac{b}{c}$ **35** and press EXE.
Note: Your calculator may display a different value of **X** at this stage. This is just the last value of **X** stored in the calculator's memory.

4. Press F6 (**SOLV**) to solve the equation.



Using these results, we are able to solve problems that involve more than one right-angled triangle.

WORKED EXAMPLE 3

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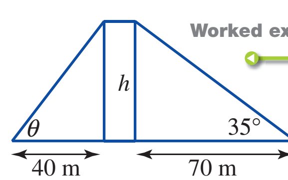
Worked example 3

Greg stands 70 m from the base of a building and measures the angle of elevation to the top of the building as being 35°. Julie is standing 40 m from the base of the building on the other side of the building as shown in the figure on the right.

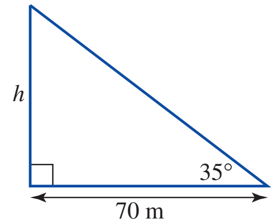


a **Calculate the height of the building, correct to 2 decimal places.**
b **Calculate the angle of elevation of the top of the building that Julie would measure, correct to the nearest degree.**

THINK

WRITE

a 1 Draw the triangle showing the angle of elevation from where Greg is standing and label the sides.



2 Choose the tangent ratio and write the formula.

$$\tan\,\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

3 Substitute for θ and the adjacent side.

$$\tan 35° = \frac{h}{70}$$

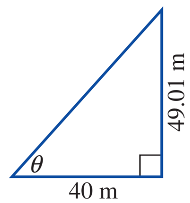
4 Make h the subject of the formula.

$$h = 70 \times \tan 35°$$

5 Calculate the value of h.

$$h = 49.01 \text{ m}$$

b 1 Draw the triangle from where Julie is standing and label the sides.



2 Choose the tangent ratio and write the formula.

$$\tan\,\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

3 Substitute for the opposite side and the adjacent side.

$$\tan\,\theta = \frac{49.01}{40}$$

4 Make θ the subject of the formula.

$$\theta = \tan^{-1}\left(\frac{49.01}{40}\right)$$

5 Calculate θ, correct to the nearest degree.

$$\theta = 51°$$

REMEMBER

1. The formulas for the three trigonometric ratios are:
   - $\sin\,\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$
   - $\cos\,\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$
   - $\tan\,\theta = \frac{\text{opposite side}}{\text{adjacent side}}$
2. To calculate the length of a side we need to be given one side length and one acute angle.
3. To calculate the size of an angle we need to be given two side lengths.
4. Many problems involve solving two or more right-angled triangles.
5. After substitution, the value of the unknown can be found using the equation solver on a graphics calculator.

EXERCISE 3A

Review of right-angled triangles

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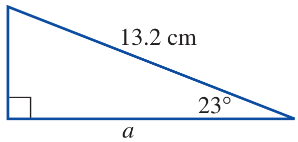
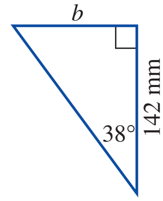
Right-angled trigonometry — finding a side length

1 WE1 Calculate the length of the side marked with the pronumerals in each of the following, correct to 1 decimal place.

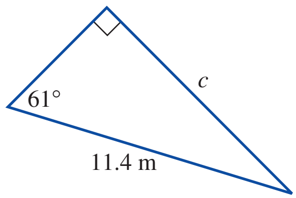
a



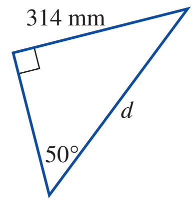
b



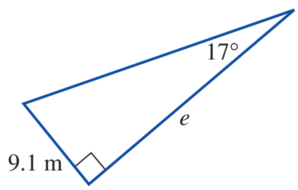
c



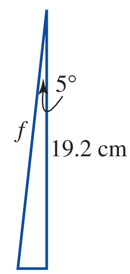
d



e



f



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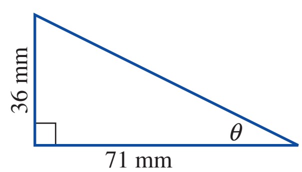
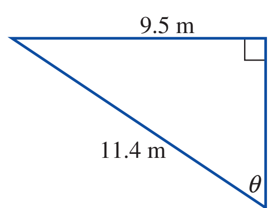
Using the inverse trigonometric ratios

2 WE2 Calculate the size of each of the angles marked with the pronumerals, correct to the nearest degree.

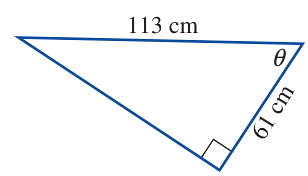
a



b



c



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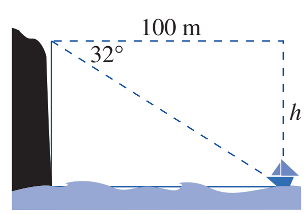
Rounding angles to the nearest degree

3 The angle of depression from the top of a cliff to a boat sailing 100 m offshore is 32°. Calculate the height of the cliff, correct to the nearest metre.



4 Andrew walks 5 km from point P to point Q. At the same time Bianca walks from P to R such that PQ is perpendicular to PR. Given that $\angle PQR = 28°$:

a draw a diagram of ΔPQR

b calculate the distance walked by Bianca, correct to the nearest metre

c calculate the distance that Andrew would need to walk in a straight line to meet Bianca, correct to the nearest metre.

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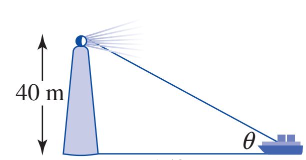
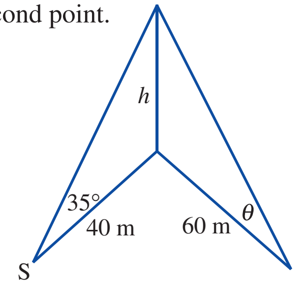
Right-angled trigonometry — finding an angle

5 A lighthouse is 40 m tall and the beacon atop the lighthouse is sighted by a ship 150 m from shore, as shown in the figure on the right. Calculate the angle of elevation at which the lighthouse is sighted from the ship, correct to the nearest degree.

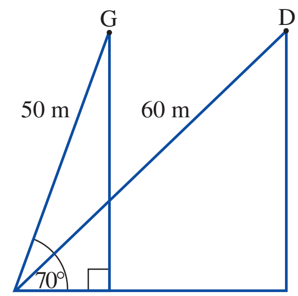


6 From a point 65 m above the ground, a second point is sighted on the ground at a distance of 239 m.
  a Draw a diagram of this situation.
  b Calculate the angle of depression from the first point to the second point.

7 WE3 Sally and Tim are both sighting the top of a building, as shown in the figure on the right. Sally is 40 m from the base of the building and sights the angle of elevation to the top of the building as 35°. Tim is 60 m from the base of the building.
  a Calculate the height of the building, correct to 2 decimal places.
  b Calculate the angle of elevation at which Tim will sight the building.



8 George and Diego are both flying a kite from the same point. George's kite is flying on 50 m of string and the string makes a 70° angle with the ground. Diego's kite is flying on a 60 m piece of string and is at the same height as George's kite, as shown in the figure on the right. Calculate the angle that the string from Diego's kite makes with the ground. Give your answer correct to the nearest degree.

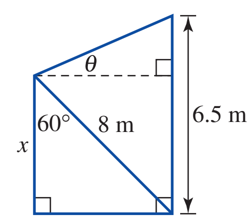


Further development

9 The shadow cast by a statue 2 metres tall is 0.6 metres. The angle of the sun to the ground is closest to:

A 17° B 18°
C 72° D 73°

10 In the diagram find x (to 1 decimal place), and θ (to the nearest degree).



11 The sun is overhead, casting a shadow of length 90 cm from a 1.75 m scarecrow, which is no longer standing upright. Determine the angle (to the nearest degree) that the scarecrow makes with the ground.

12 A kite is hovering in strong winds, 10 m vertically above the ground. It is being held in place by a taut 12-m length of rope from the kite to the ground. Find the angle (to the nearest degree) that the rope makes with the ground.

13 A ramp joins two points, A and B. The horizontal distance between A and B is 1.4 m, and B is 30 cm vertically above the level of A.
  a Find the length of the ramp (in metres to 2 decimal places).
  b Find (to the nearest degree) the angle that the ramp makes with the horizontal.

14 A chairlift follows a direct line from a mountain peak (altitude 1400 m) to a ridge (altitude 960 m). If the horizontal distance between the peak and the ridge is 510 m, find the angle of descent (to the nearest degree) from one to the other.

3B Bearings

A bearing is an angle used to describe direction. Bearings are used in navigation and are a common application of trigonometry to practical situations. We can therefore apply our trigonometrical formulas to make calculations based upon these bearings. There are two types of bearing that we need to be able to work with: compass bearings and true bearings.

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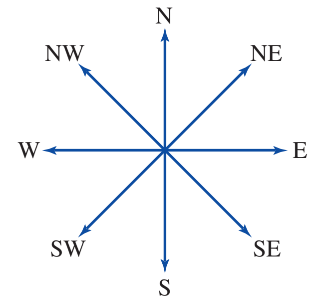
Navigation and specifications of locations

Compass bearings

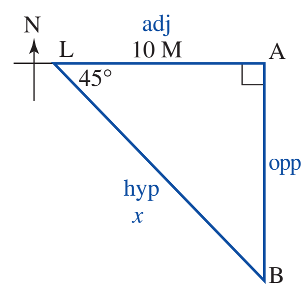
Compass bearings use the four points of the compass. With compass bearings there are four main directions: north, south, east and west. In between each of these main directions there are four others: north-east, south-east, south-west and north-west. Each of these directions is at 45° to two of the four main directions.

Trigonometry can then be used to solve problems about distances and angles using these eight basic directions.



WORKED EXAMPLE 4

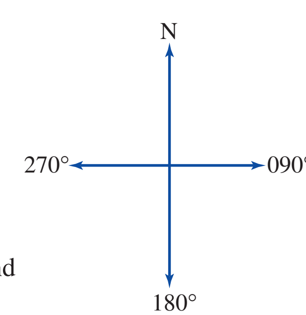
A ship (A) is 10 nautical miles due east of a lighthouse. A second ship (B) bears SE of the lighthouse and is due south of the first ship. Calculate the distance of the second ship from the lighthouse, correct to 1 decimal place.

THINK	WRITE
1 Draw a diagram labelling the sides of the triangle.	
2 Choose the cosine ratio and write the formula.	$\cos\,\theta = \dfrac{\text{adj}}{\text{hyp}}$
3 Substitute for θ and the adjacent side.	$\cos\,45° = \dfrac{10}{x}$
4 Make x the subject of the equation.	$x\cos\,45° = 10$ $x = \dfrac{10}{\cos\,42°}$
5 Calculate the value of x, correct to 1 decimal place.	$= 14.1$ M
6 Give a written answer.	The second ship is 14.1 nautical miles from the lighthouse.

These eight compass points do not allow us to make calculations about more precise directions. For this reason an alternative method of describing bearings is needed for any direction other than these basic eight points.

True bearings

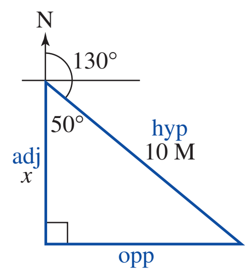
A true bearing is an angle measured from north in a clockwise direction. As there are 360° in a revolution, all true bearings are represented as a three-digit number between 000° and 360°. For example, east is at a bearing of 090°, south has a bearing of 180° and west 270°.



When given information about a bearing, we can solve problems using trigonometry by constructing a right-angled triangle. As most questions involving bearings are in problem form, a diagram is necessary to solve the problem and an answer in words should be given.

WORKED EXAMPLE 5

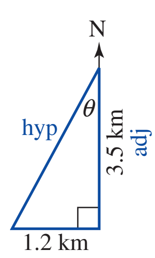
A ship sails on a bearing of 130° for a distance of 10 nautical miles. Calculate how far south of its starting point the ship is, correct to 2 decimal places.

THINK	WRITE
1 Draw a diagram completing a right-angled triangle and label the sides.	
2 Choose the cosine ratio and write the formula.	$\cos\,\theta = \dfrac{\text{adj}}{\text{hyp}}$
3 Substitute for θ and the hypotenuse.	$\cos 50° = \dfrac{x}{10}$
4 Make x the subject of the equation.	$x = 10\cos 50°$
5 Calculate.	$x = 6.43$ M
6 Give a written answer.	The ship is 6.43 nautical miles south of its starting point.

We can also use our methods of calculating angles to make calculations about bearings. After solving the right-angled triangle, however, we need to provide the answer as a bearing.

WORKED EXAMPLE 6

On a hike Lisa walked south for 3.5 km and then turned west for 1.2 km. Calculate Lisa's bearing from her starting point.

THINK	WRITE
1 Draw a diagram and label the sides of the triangle.	
2 Choose the tangent ratio and write the formula.	$\tan\,\theta = \dfrac{\text{opp}}{\text{adj}}$
3 Substitute for the opposite and adjacent sides and simplify.	$\tan\,\theta = \dfrac{1.2}{3.5}$ $= 0.3429$

4	Make θ the subject of the equation.	$\theta = \tan^{-1}(0.3429)$
5	Calculate θ.	$= 19°$
6	From the diagram we can see the angle lies between south and west. South has a bearing of 180°, and so we must add 19° to 180° to calculate the true bearing.	Bearing = 180° + 19° = 199°
7	Give a written answer.	Lisa is at a bearing of 199° from her starting point.

WORKED EXAMPLE 7

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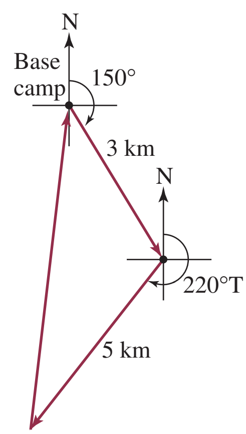
Worked example 7

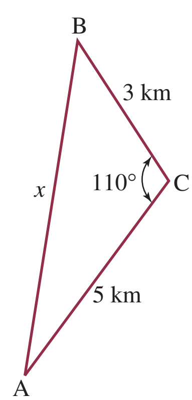
Soldiers on a reconnaissance set off on a return journey from their base camp. The journey consists of three legs. The first leg is on a bearing of 150°T for 3 km; the second is on a bearing of 220°T for 5 km. Find the direction (to the nearest minute) and distance (correct to 2 decimal places) of the third leg by which the group returns to its base camp.

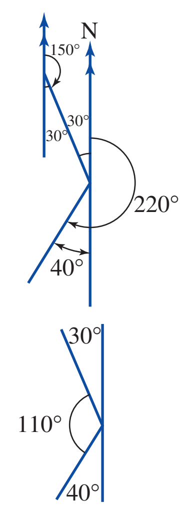
THINK | **WRITE/DRAW**

1 Draw a diagram of the journey and indicate or superimpose a suitable triangle.



2 Identify the side of the triangle to be found. Redraw a simple triangle with the most important information provided.





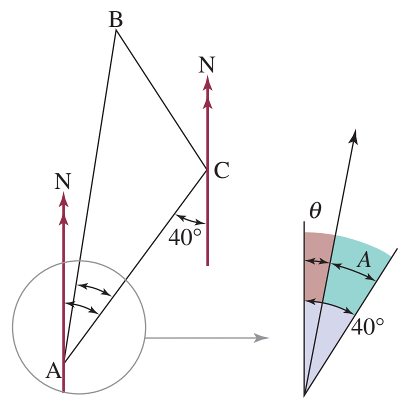
3 Identify that the problem requires the use of the cosine rule, as you are given two sides and the angle in between.

$a = 3$ km $\quad b = 5$ km $\quad C = 110°\quad c = x$ km

4 Substitute the known values into the cosine rule and evaluate.

$$c^2 = a^2 + b^2 - 2ab \times \cos(C)$$
$$x^2 = 3^2 + 5^2 - 2 \times 3 \times 5 \times \cos(110°)$$
$$x^2 = 44.260\,604$$
$$x = \sqrt{44.260\,604}$$
$$= 6.65$$

5 For direction, we need to find the angle between the direction of the second and third legs using the sine or cosine rules.



$a = 3\quad b = 5\quad c = 6.65$ or $\sqrt{44.260\,604}$

6 Substitute the known values into the rearranged cosine rule.

Note: Use the most accurate form of the length of side *c*.

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2 \times b \times c}$$
$$\cos(A) = \frac{5^2 + 44.260\,604 - 3^2}{2 \times 5 \times \sqrt{44.260\,604}}$$
$$\cos(A) = 0.9058$$
$$A = 25.07°$$
$$= 25°4'$$

7 Calculate the angle of the turn from the north bearing.

$$\theta = 40° - 25°4'$$
$$= 14°56'$$

Bearing is N14°56′E.

8 Write the answer in correct units and to the required level of accuracy.

The distance covered in the final leg is 6.65 km, correct to 2 decimal places, on a bearing of N14°56′E, correct to the nearest minute.

REMEMBER

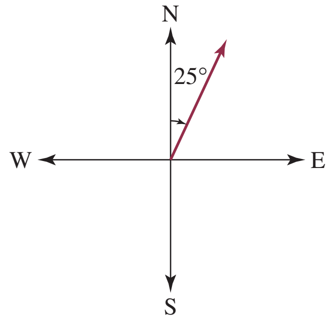
1. Bearings are used to describe a direction. We have used two types of bearings.
   - Compass bearings use the four main points of the compass, north, south, east and west, as well as the four middle directions, north-east, north-west, south-east and south-west.
   - True bearings describe more specific direction by using a three-digit angle, which is measured from north in a clockwise direction.
2. Bearing questions are usually given in written form so you will need to draw a diagram to extract all the information from the question.

3. Read carefully to see if the question is asking you to find a side or an angle.
4. Always give a written answer to worded questions.
5. Use 1 M = 1.852 km to convert between nautical miles and kilometres.

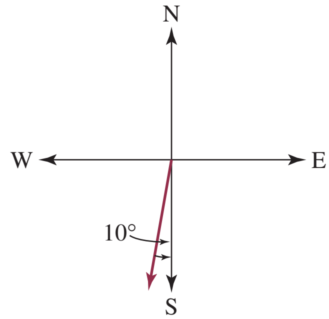
EXERCISE
3B Bearings

1 Specify the following directions as compass bearings.

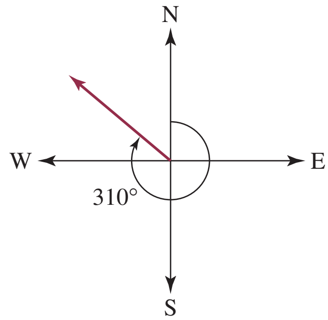
a



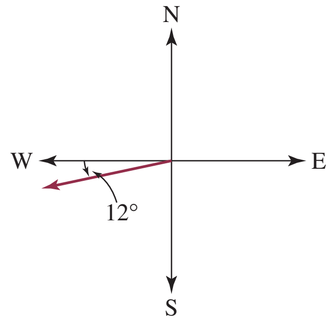
b



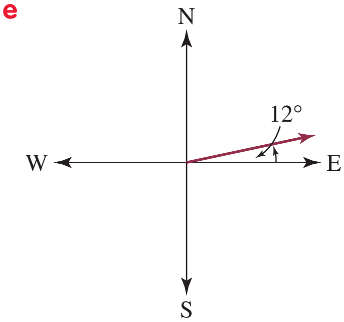
c



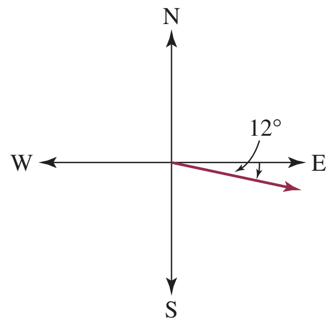
d



e

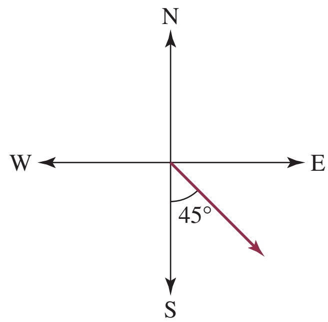


f



2 Specify the following directions as true bearings.

a



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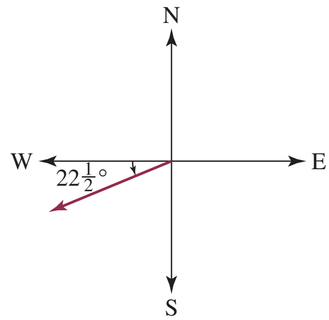
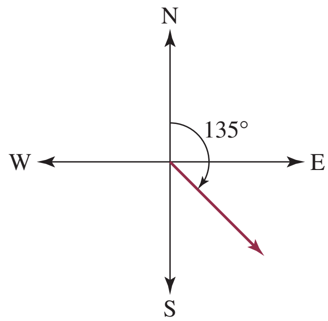


c



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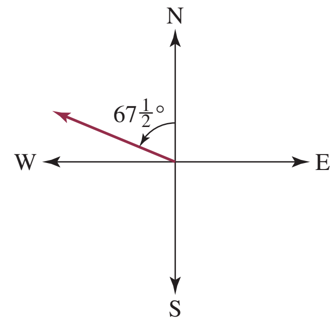
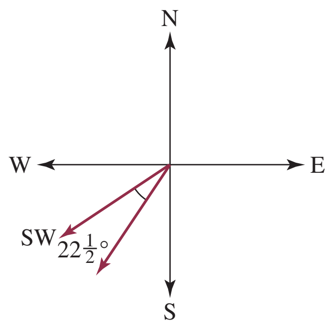
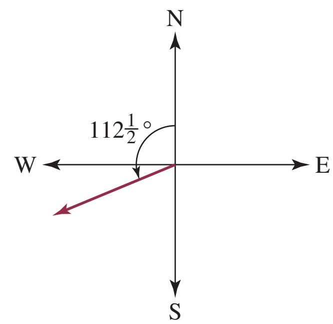


e



f



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Converting nautical miles to kilometres

3 WE4 A road runs due north. A hiker leaves the road and walks for 4.2 km in a NW direction.
  a Draw a diagram of this situation.
  b How far due east must the hiker walk to get back to the road? (Give your answer correct to 3 decimal places.)

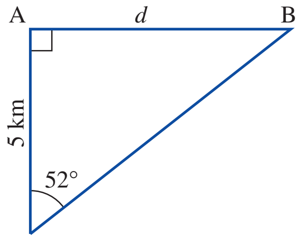
4 A driver heads due south for 34 km, then turns left and drives until he is SE of his starting point.
  a Draw a diagram to show the driver's journey.
  b Calculate the distance the driver travelled in an easterly direction from his starting point.

5 Two boats, A and B, sail from a port. A heads due west, while B heads NW for a distance of 43 nautical miles, where it drops anchor. Boat A drops anchor due south of boat B.
  a Draw a diagram showing the positions of boats A and B.
  b Calculate the distance between boats A and B in nautical miles, correct to 1 decimal place.
  c Calculate the distance in kilometres between A and B.

6 MC A true bearing of 315° is equivalent to a compass bearing of:
  A NE
  B NW
  C SE
  D SW

7 MC A compass bearing of SE is equivalent to a true bearing of:
  A 045°
  B 135°
  C 225°
  D 315°

8 WE5 Two hikers, Adrian and Bertrand, set out on a walk. Adrian walks 5 km due north to a point, A, and Bertrand walks on a bearing of 052° to a point, B. Bertrand lets off a flare and Adrian notices Bertrand is now due east of him, as shown in the diagram on the right. Calculate the distance between the two hikers, correct to 1 decimal place.



9 A yacht sights a lighthouse on a bearing of 060°. After sailing another eight nautical miles due north, the yacht is due west of the lighthouse.

a Draw a diagram of this situation.

b Calculate the distance from the yacht to the lighthouse when it is due west of it (correct to 1 decimal place).

10 An aeroplane takes off from an airport and flies on a bearing of 220° for a distance of 570 km. Calculate how far south of the airport the aeroplane is (correct to the nearest kilometre).

11 A camping ground is due east of a car park. Eden and Jeff walk 3.8 km due south from the camping ground until the car park is on a bearing of 290°.

a Draw a diagram showing the car park, the camping ground, and Eden and Jeff's position.

b Calculate the distance Eden and Jeff need to walk directly back to the car park, correct to 1 decimal place.

12 MC A ship is on a bearing of 070° from a lighthouse. The bearing of the lighthouse from the ship will be:

A 070° B 160° C 200° D 250°

13 MC A camping ground is SW of a car park. The bearing of the car park from the camping ground will be:

A NE B NW C SE D SW

14 WE6 A search party leaves its base and head 4 km due west before turning south for 3.5 km.

a Draw a diagram of this situation.

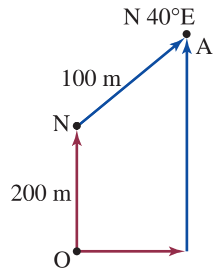
b Calculate the true bearing of the search party from its base, correct to the nearest degree.

15 WE7 A ship is two nautical miles due west of a harbour. A yacht that sails 6.5 nautical miles from that harbour is due north of the ship. Calculate the true bearing (correct to the nearest degree) of the course on which the yacht sails from the harbour.

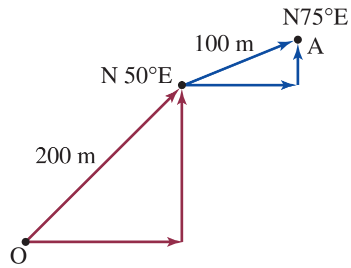
Further development

16 For each of the following, find how far north/south and east/west position A is from position O.

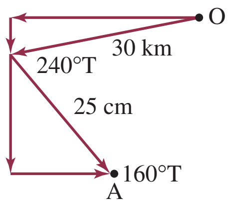
a



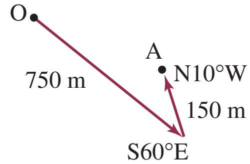
b



c



d



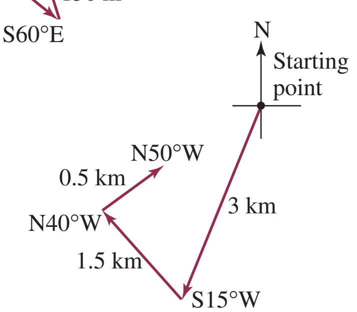
17 The distances covered in a yachting regatta are shown in the diagram. Find (to the nearest metre):

a how far south the yacht is from the starting point

b how far west the yacht is from the starting point

c the distance from the starting point

d the direction of the final leg to return to the starting point.

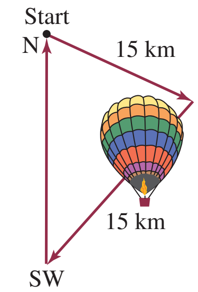


18 Captain Cook sailed from Cook Island due north for 100 km. He then changed direction and sailed west to a deserted island that is NW of Cook island.

a How far was Captain Cook from Cook Island?

b What distance could have been saved if Captain Cook had sailed directly?

19 A diagram representing the journey by a hot-air balloon is shown. The balloonist recorded the first leg of the journey as 15 km SE. The second leg was also 15 km. Find the distance for the final leg of the balloonist's journey.



20 A golfer is aiming for a hole that is 190 metres on a bearing of 220° from the tee. She hits the ball 8° off centre to a point that is on a bearing of 310° from the hole. How far does she have to hit the ball to reach the hole?

Trigonometric ratios for obtuse angles

Many non-right-angled triangles have one obtuse angle. In the following sections we will be solving non-right-angled triangles and will need to investigate the trigonometric ratios for obtuse angles.

1 Use your calculator to give each of the following, correct to 3 decimal places.

a sin 100°	**b** cos 100°	**c** tan 100°
d sin 135°	**e** cos 135°	**f** tan 135°
g sin 179°	**h** cos 179°	**i** tan 179°

2 Which of the answers to question **1** are positive and which are negative?

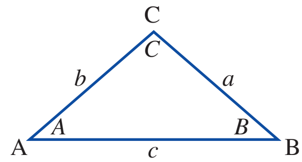
3 Calculate the sine, cosine and tangent of several other obtuse angles and see if the established pattern continues.

4 Can you develop a rule for the sign of trigonometric ratios of obtuse angles?

3C Using the sine rule to find side lengths

Finding side lengths

The trigonometry we have studied so far has been applicable to only right-angled triangles. The **sine rule** allows us to calculate the lengths of sides and the size of angles in non-right-angled triangles. Consider the triangle drawn on the right.



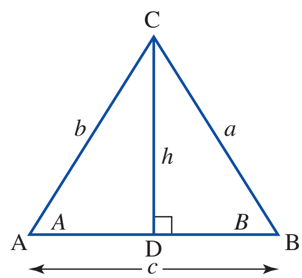
The sine rule states that in any triangle, ABC, the ratio of each side to the sine of its opposite angle will be equal.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

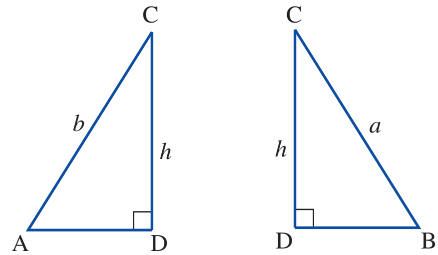
Derivation of the sine rule

A, B and C represent the three angles in the triangle ABC and a, b and c represent the three sides, remembering that each side is named with the lower-case letter of the opposite vertex.

Construct a line from C to a point, D, perpendicular to AB. CD is the perpendicular height of the triangle, h.



Now consider ΔACD and ΔBCD separately.



Use the formula for the sine ratio:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \sin\theta = \frac{\text{opp}}{\text{hyp}}$$

$$\sin A = \frac{h}{b} \qquad \sin B = \frac{h}{a}$$

$$h = b \sin A \qquad h = a \sin B$$

We are now able to equate these two expressions for h.

$$a \sin B = b \sin A$$

Dividing both sides by $\sin A \sin B$ we get:

$$\frac{a \sin B}{\sin A \sin B} = \frac{b \sin A}{\sin A \sin B}$$

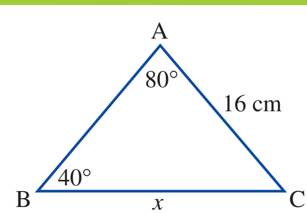
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Similarly, we are able to show that each of these is also equal to $\frac{c}{\sin C}$. Try it!

This formula allows us to calculate the length of a side in any triangle if we are given the length of one other side and two angles. When using the formula we need to use only two parts of it.

WORKED EXAMPLE 8

Calculate the length of the side marked x in the triangle on the right, correct to 1 decimal place.



THINK | **WRITE**

Method 1: Technology-free

1 Write the formula.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

2 Substitute $a = x$, $b = 16$, $A = 80°$ and $B = 40°$.

$$\frac{x}{\sin 80°} = \frac{16}{\sin 40°}$$

3 Make x the subject of the equation by multiplying by $\sin 80°$.

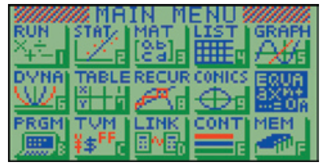
$$x = \frac{16 \sin 80°}{\sin 40°}$$

4 Calculate.

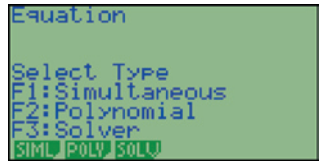
$$x = 24.5 \text{ cm}$$

Method 2: Technology-enabled

1. From the **MENU** select **EQUA**.

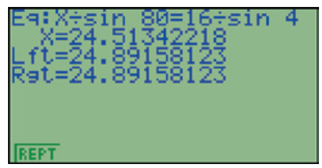


2. Press F3 (**SOLV**).



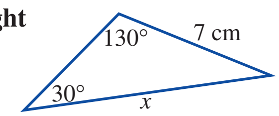
3. Delete any existing equation, enter the equation **X ÷ sin 80 = 16 ÷ sin 40**, and then press EXE.
Note: Your calculator may display a different value of **X** at this stage. This is just the last value of **X** stored in the calculator's memory.

4. Press F6 (**SOLV**) to solve the equation.



WORKED EXAMPLE 9

Find the unknown length, x cm, in the triangle at right (to 1 decimal place).



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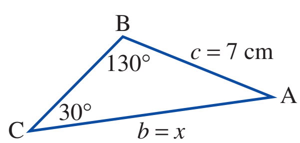
Tutorial
int-0465
Worked example 9

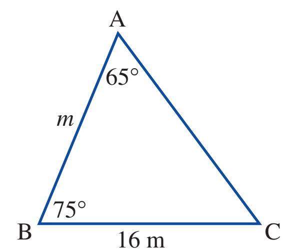
THINK	WRITE/DISPLAY
1. Draw the triangle. Assume it is non-right-angled.	
2. Label the triangle appropriately for the sine rule.	
3. Confirm that it is the sine rule that can be used as you have the angle opposite to the unknown side and a known $\frac{\text{side}}{\text{angle}}$ ratio.	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$ $b = x \qquad B = 130°$ $c = 7\text{ cm} \qquad C = 30°$
4. Substitute known values into the two ratios.	$\frac{x}{\sin(130°)} = \frac{7}{\sin(30°)}$
5. Isolate x and evaluate.	$x = \frac{7 \times \sin(130°)}{\sin(30°)}$ $x = 10.7246$ $x = 10.7$
6. Write the answer.	The unknown length is 10.7 cm, correct to 1 decimal place.

Note: Some questions may ask for you to give the answer in a form other than a number and as such the graphics calculator method can not be used. For example, the question above could be worded to, say, show $x = \dfrac{16 \sin 80°}{\sin 40°}$, in which case you must manipulate the equation to arrive at the desired expression.

To use the sine rule we need to know the angle opposite the side we are finding and the angle opposite the side we are given. In some cases these are not the angles we are given. In such cases we need to use the fact that the angles in a triangle add to 180° to calculate the required angle.

WORKED EXAMPLE 10

Calculate the length of the side labelled *m* in the figure on the right, correct to 4 significant figures.



THINK

WRITE

1. Calculate the size of angle C.

$C = 180° - 65° - 75°$
$= 40°$

2. Write the formula.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

3. Substitute $a = 16$, $c = m$, $A = 65°$ and $C = 40°$.

$$\frac{16}{\sin 65°} = \frac{m}{\sin 40°}$$

4. Make m the subject of the equation.

$$m = \frac{16 \sin 40°}{\sin 65°}$$

5. Calculate.

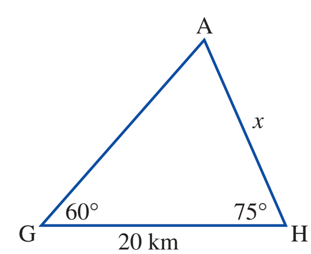
$= 11.35$ m

As mentioned in the previous investigation, we need to apply the sine rule to obtuse-angled triangles. In such examples the method used is exactly the same with the substitution of an obtuse angle.

Using the sine rule allows us to solve a number of more complex problems. As with our earlier trigonometry problems, we begin each with a diagram and give a written answer to each.

WORKED EXAMPLE 11

Georg looks south and observes an aeroplane at an angle of elevation of 60°. Henrietta is 20 km south of where Georg is and she faces north to see the aeroplane at an angle of elevation of 75°. Calculate the distance of the aeroplane from Henrietta's observation point, to the nearest metre.



THINK	WRITE
1 Calculate the size of ∠GAH.	$A = 180° - 60° - 75°$ $= 45°$
2 Write the formula.	$\dfrac{g}{\sin G} = \dfrac{a}{\sin A}$
3 Substitute $g = x$, $a = 20$, $G = 60°$ and $H = 75°$.	$\dfrac{x}{\sin 60°} = \dfrac{20}{\sin 45°}$
4 Make x the subject.	$x = \dfrac{20 \sin 60°}{\sin 45°}$
5 Calculate.	$x = 24.495$ km
6 Give a written answer.	The distance of the aeroplane from Henrietta's observation point is 24.495 km.

REMEMBER

1. The sine rule formula is $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.
2. The sine rule is used to find a side in any triangle when we are given the length of one other side and two angles.
3. We need to use only two parts of the sine rule formula.
4. For written problems, begin by drawing a diagram and finish by giving a written answer.
5. You can use the equation solver on a graphics calculator to find the value of the unknown after substituting into the formula.

EXERCISE 3C Using the sine rule to find side lengths

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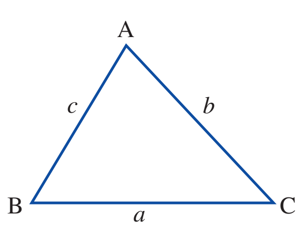
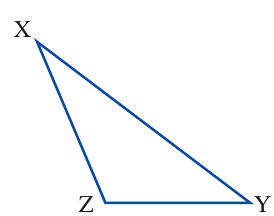
Angle sum of a triangle

1 Write down the sine rule formula as it applies to each of the triangles below.

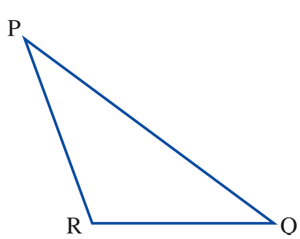
a



b



c



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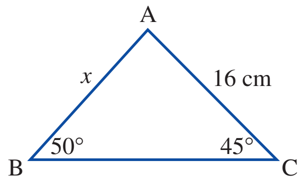
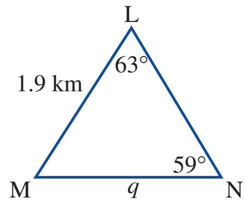
Solving fractional equations

2 WE8, 9 Use the sine rule to calculate the length of the side marked with the pronumeral in each of the following, correct to 3 significant figures.

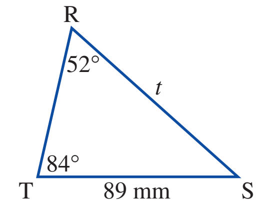
a



b

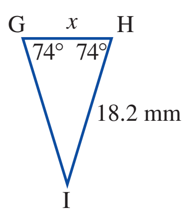


c

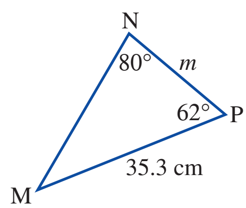


3 WE10 In each of the following, use the sine rule to calculate the length of the side marked with the pronumeral, correct to 1 decimal place, by first finding the size of the third angle.

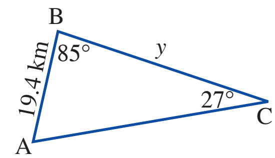
a



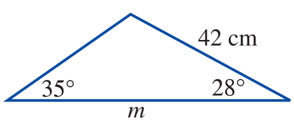
b



c



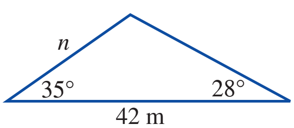
4 MC Look at the figure drawn on the right. Which of the following expressions gives the value of m?



A $m = \dfrac{42 \sin 117°}{\sin 28°}$ **B** $m = \dfrac{42 \sin 117°}{\sin 35°}$

C $m = \dfrac{42 \sin 28°}{\sin 117°}$ **D** $m = \dfrac{42 \sin 35°}{\sin 117°}$

5 MC Look at the figure drawn on the right. Which of the following expressions gives the value of n?



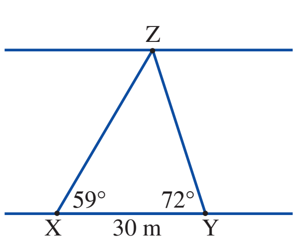
A $n = \dfrac{42 \sin 117°}{\sin 28°}$ **B** $n = \dfrac{42 \sin 117°}{\sin 35°}$

C $n = \dfrac{42 \sin 28°}{\sin 117°}$ **D** $n = \dfrac{42 \sin 35°}{\sin 117°}$

6 ABC is a triangle in which BC = 9 cm, ∠BAC = 54° and ∠ACB = 62°. Calculate the length of side AB, correct to 1 decimal place.

7 XYZ is a triangle in which $y = 19.2$ m, ∠XYZ = 42° and ∠XZY = 28°. Calculate x, correct to 3 significant figures.

8 WE11 X and Y are two trees, 30 m apart on one side of a river. Z is a tree on the opposite side of the river, as shown in the diagram below.



It is found that ∠XYZ = 72° and ∠YXZ = 59°. Calculate the distance XZ, correct to 1 decimal place.

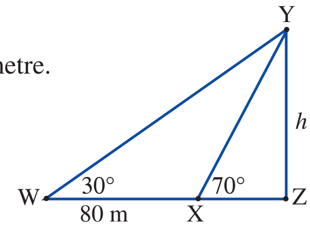
9 From a point, M, the angle of elevation to the top of a building, B, is 34°. From a point, N, 20 m closer to the building, the angle of elevation is 49°.

a Draw a diagram of this situation.

b Calculate the distance NB, correct to 1 decimal place.

c Calculate the height of the building, correct to the nearest metre.

10 Look at the figure on the right.



a Show that XY can be given by the expression $\dfrac{80 \sin 30°}{\sin 40°}$.

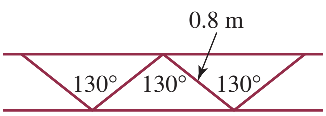
b Show that h can be found using the expression $\dfrac{80 \sin 30° \ \sin 70°}{\sin 40°}$.

c Calculate h, correct to 1 decimal place.

Further development

11 Steel trusses are used to support a heavy gate at the entrance to a shipping yard.

The struts in the truss shown are each made from 0.8 m steel lengths and are welded at the contact points with the upper and lower sections of the truss.



a On the lower section of the truss, what is the distance (to the nearest centimetre) between each pair of consecutive welds?

b What is the height (to the nearest centimetre) of the truss?

12 A scenic flight leaves Town A and flies west of north for the 80-km direct journey to Town B. At Town B the plane turns 92° to the right to fly east of north to Town C. From here the plane turns 129° to the right and flies the 103 km straight back to Town A. Find the distance (to the nearest km) of the direct flight from Town B to Town C.

13 a Two lighthouses are 17 kilometres apart on an east–west line. From lighthouse A, a ship is seen on a bearing of 130°. From lighthouse B, the same ship is spotted on a bearing of S20°W. Which lighthouse is the ship closer to? How far is that lighthouse from the ship?

b Two lighthouses are 25 kilometres apart on a south–north line. From lighthouse A, a ship is reported on a bearing of 082°T. The same ship is detected from lighthouse B on a bearing of 165°T. Which lighthouse is closer to the ship and how far is that lighthouse from the ship?

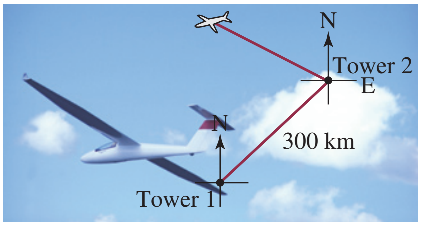
c Two fire-spotting towers are 33 km apart on an east–west line. From Tower A a fire is spotted on a bearing of N63°E, while from Tower B the same fire is spotted on a bearing of 290°T. How far away from the nearer tower is the fire? Which tower is this?

14 Two lighthouses are 25 km apart on a north–south line. The northern lighthouse spots a ship on a bearing of S60°E. The southern lighthouse spots the same ship on a bearing of 050°T.

a Find the distance from the northern lighthouse to the ship.

b Find the distance from the southern lighthouse to the ship.

15 A light aircraft has strayed into a major air corridor. It has been detected by two air traffic control towers.
Tower 1 has the light aircraft on a bearing of 315°T.
Tower 2 has the light aircraft on a bearing of north.
The two towers are 300 kilometres apart on a NE line as shown. How far is the light plane from each tower?



3D Using the sine rule to find angles

Finding angles

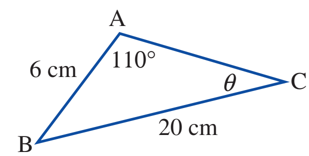
Using the sine rule result, we are able to calculate angle sizes as well. To do this, we need to be given the length of two sides and the angle opposite one of them. For simplicity, in solving the triangle we invert the sine rule formula when we are using it to find an angle. The formula is written:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Your formula sheet has the sine rule to find a side length. You need to invert this formula when finding an angle. As with finding side lengths, we use only two parts of the formula.

WORKED EXAMPLE 12

Find the size of the angle, θ, in the figure on the right, correct to the nearest degree.



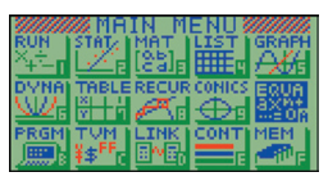
THINK / **WRITE**

Method 1: Technology-free

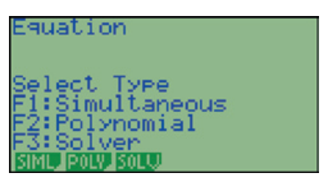
1. Write the formula. $$\frac{\sin A}{a} = \frac{\sin C}{c}$$
2. Substitute $A = 110°$, $C = \theta$, $a = 20$ and $c = 6$. $$\frac{\sin 110°}{20} = \frac{\sin \theta}{6}$$
3. Make $\sin \theta$ the subject of the equation. $$\sin \theta = \frac{6 \sin 110°}{20}$$
4. Calculate a value for $\sin \theta$. $$\sin \theta = 0.2819$$
5. Calculate $\sin^{-1}(0.2819)$ to find θ. $$\theta = 16°$$

Method 2: Technology-enabled

1. From the **MENU** select **EQUA**.

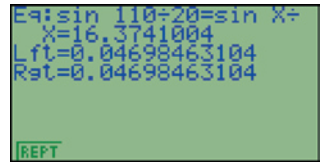


2. Press F3 (**SOLV**).



3. Delete any existing equation, enter the equation **sin 110° ÷ 20 = sin X ÷ 6** and press EXE.
Note: Your calculator may display a different value of **X** at this stage. This is just the last value of **X** stored in the calculators memory.

4 Press F6 (**SOLV**) to solve the equation.



Note: When using the graphics calculator, you do not need to remember to invert the sine rule. If you enter 20 ÷ sin 110 = 6 sin *x*, the graphics calculator will still solve the equation.

As with finding side lengths, some questions will be problems that require you to draw a diagram to extract the required information and then write the answer.

WORKED EXAMPLE 13

From a point, P, a ship (S) is sighted 12.4 km from P on a bearing of 137°. A point, Q, is due south of P and is a distance of 31.2 km from the ship. Calculate the bearing of the ship from Q, correct to the nearest degree.

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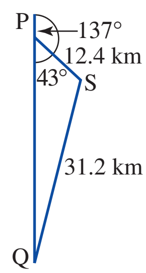
Worked example 13

THINK / **WRITE**

1 Draw a diagram.



2 Write the formula.

$$\frac{\sin Q}{q} = \frac{\sin P}{p}$$

3 Substitute for p, q and P.

$$\frac{\sin Q}{12.4} = \frac{\sin 43°}{31.2}$$

4 Make $\sin Q$ the subject.

$$\sin Q = \frac{12.4 \sin 43°}{31.2}$$

5 Calculate a value for $\sin Q$.

$$\sin Q = 0.271$$

6 Calculate $\sin^{-1}(0.271)$ to find Q.

$$Q = 16°$$

7 Give a written answer.

The bearing of the ship from Q is 016°.

REMEMBER

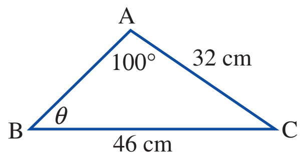
1. The sine rule formula for finding an angle is $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.
2. The formula sheet gives the sine rule in the form used to find a side. You have to invert the formula when finding angles.
3. We can use this formula when we are given two sides and the angle opposite one of them.
4. Worded questions should begin with a diagram and finish with a written answer.

EXERCISE

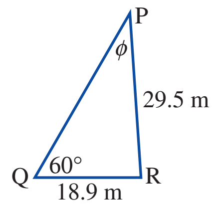
3D Using the sine rule to find angles

1 WE12 Find the size of the angle marked with a pronumeral in each of the following, correct to the nearest degree.

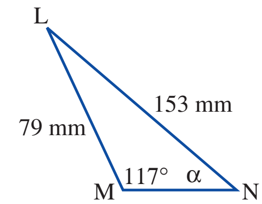
a



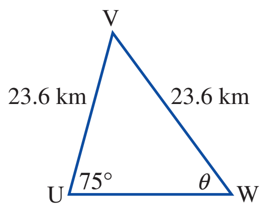
b



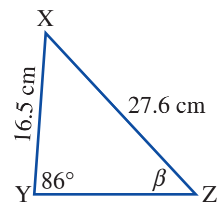
c



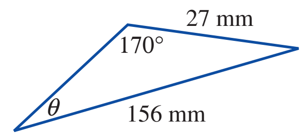
d



e



f



2 MC Which of the statements below give the correct value for $\sin\theta$?

A $\sin\theta = \dfrac{13\sin 36°}{7}$

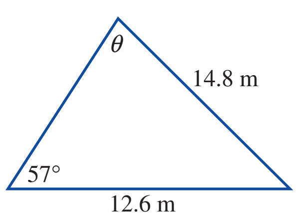
B $\sin\theta = \dfrac{7\sin 36°}{13}$

C $\sin\theta = \dfrac{36\sin 13°}{7}$

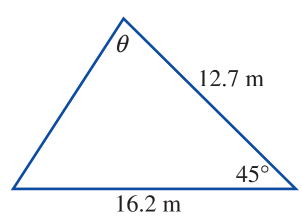
D $\sin\theta = \dfrac{7\sin 13°}{36}$

3 MC In which of the triangles below is the information insufficient to use the sine rule?

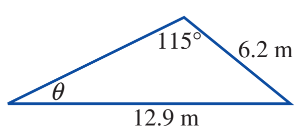
A



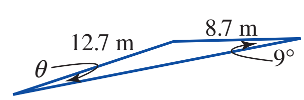
B



C

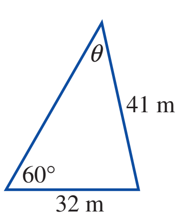


D

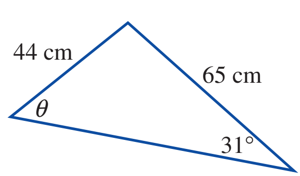


4 In Questions **a**–**c** find the size of the angle marked θ, correct to the nearest degree.

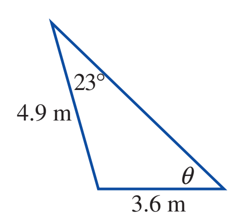
a



b



c



5 In ΔPQR, $q = 12$ cm, $r = 16$ cm and $\angle$PRQ = 56°. Find the size of $\angle$PQR, correct to the nearest degree.

6 In ΔKLM, LM = 4.2 m, KL = 5.6 m and $\angle$KML = 27°. Find the size of $\angle$LKM, correct to the nearest degree.

7 WE13 A, B and C are three towns marked on a map. Judy calculates that the distance between A and B is 45 km and the distance between B and C is 32 km. ∠CAB is 45°. Calculate ∠ACB, correct to the nearest degree.

8 A surveyor marks three points X, Y and Z in the ground. The surveyor measures XY to be 13.7 m and XZ to be 14.2 m. ∠XYZ is 60°.
a Calculate ∠XZY to the nearest degree.
b Calculate ∠YXZ to the nearest degree.

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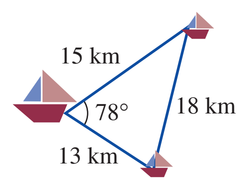
9 Two wires support a flagpole. The first wire is 8 m long and makes a 65° angle with the ground. The second wire is 9 m long. Find the angle that the second wire makes with the ground.

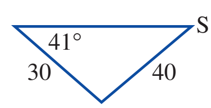
10 Construct a suitable triangle from the following instructions and find all unknown sides and angles. The smallest side is 17 cm and one of the other sides is 25 cm. The smallest angle is 32°.

11 A yacht sails the three-leg course shown.
The smallest angle between any two legs within the course, to the nearest degree, is:



A 34°
B 55°
C 45°
D 78°

12 The correct expression for angle S in the given triangle is:



A $\sin^{-1}\left(\dfrac{40 \times \sin 41°}{30}\right)$

B $\sin^{-1}\left(\dfrac{30 \times \sin 41°}{40}\right)$

C $\sin^{-1}\left(\dfrac{41 \times \sin 41°}{30}\right)$

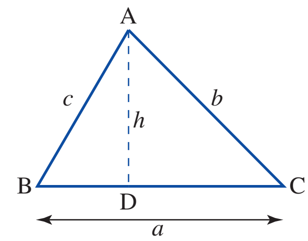
D $\sin^{-1}\left(\dfrac{30}{40 \times \sin 41°}\right)$

13 Two army camps A and B are on the same east–west line. Radio tower T is located 20 km from camp A, SE of camp A. The tower is a distance of 15 km from camp B. Find the bearing of the radio tower, T, from camp B.

14 Ben is planning to hike in the mountains out of the snow season. From a position in front of the ski lodge, Ben can see the chairlift station and the start of the ski run in the distance. He notes that in moving his eye from the ski lodge station to the start of the run, the angle is 34°. Ben then walks in a straight line to the chairlift station, a distance of 365 m, turns and walks the 230 m straight line distance to the start of the ski run. From here, what angle (to the nearest degree) would he note between the ski lodge and the chairlift station?

15 A golfer is teeing off on the 1st hole. The distance and direction to the green is 410 metres on a bearing of 190°. If the tee shot of the player was 250 metres on a bearing of 220°, and if the distance between where her tee shot lands and the green is 230.36 m, what direction should she aim for? (Give the direction to the nearest degree.)

3E Area of a triangle



You should be familiar with finding the area of a triangle using the formula Area $= \frac{1}{2}bh$. In this formula, b is the base of the triangle and h is the perpendicular height. This formula can't be used in triangles where we do not know the perpendicular height. Trigonometry allows us to find the area of such triangles when we are given the length of two sides and the included angle.

Consider the triangle drawn on the right. In this triangle:

$$\text{Area} = \frac{1}{2}ah \qquad [1]$$

(a = base of triangle, h = height)

Now consider ΔACD. Since this triangle is right angled:

$$\sin C = \frac{\text{opp}}{\text{hyp}}$$

$$\sin C = \frac{h}{b}$$

$$h = b \sin C$$

Substituting for h in [1]:

$$\text{Area} = \frac{1}{2}ab\sin C$$

This becomes the formula for the area of a triangle. There are three equivalent formulas for the area of a triangle.

$$\textbf{Area} = \frac{1}{2}ab\sin C$$

$$\textbf{Area} = \frac{1}{2}ac\sin B$$

$$\textbf{Area} = \frac{1}{2}bc\sin A$$

The formula sheet gives the first version of this formula only. The others are an adaptation of the same rule. These formulas allow us to find the area of any triangle where we are given the length of two sides and the included angle. The included angle is the angle between the two given sides. The formula chosen should be the one that uses the angle you have been given.

WORKED EXAMPLE 14

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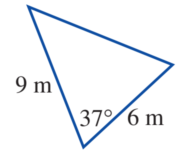
Worked example 14

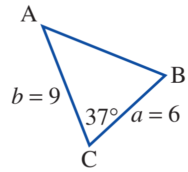
Find the area of the triangle at below (to 2 decimal places).



THINK

1. Identify the shape as a triangle with two known sides and the angle in between.

WRITE/DISPLAY



2. Identify and write down the values of the two sides, a and b, and the angle in between them, C.

$a = 6$
$b = 9$
$C = 37°$

3. Identify the appropriate formula and substitute the known values into it.

$$\begin{aligned}\text{Area}_{\text{triangle}} &= \tfrac{1}{2}\,ab \sin (C) \\ &= \tfrac{1}{2} \times 6 \times 9 \times \sin (37°) \\ &= 16.249\end{aligned}$$

4. Write the answer in correct units.

The area of the triangle is 16.25 m^2, correct to 2 decimal places.

As with all other trigonometry we can use this formula to solve practical problems.

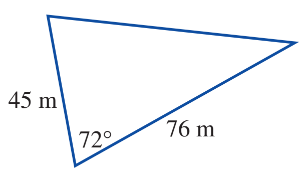
WORKED EXAMPLE 15

Two paths diverge at an angle of 72°. The paths' lengths are 45 m and 76 m respectively. Calculate the area between the two paths, correct to the nearest square metre.

THINK

1. Draw a diagram.
2. Write the formula.
3. Substitute $a = 45$, $b = 76$ and $C = 72°$.
4. Calculate.
5. Give a written answer.

WRITE



$$\begin{aligned}\text{Area} &= \tfrac{1}{2}ab \sin C \\ &= \tfrac{1}{2} \times 45 \times 76 \times \sin 72° \\ &= 1626 \text{ m}^2\end{aligned}$$

The area between the paths is 1626 m^2.

REMEMBER

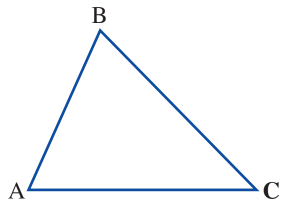
1. The area of a triangle can be found when you are given the length of two sides and an included angle.
2. The formulas to use are:
$$\text{Area} = \tfrac{1}{2}ab \sin C$$
$$\text{Area} = \tfrac{1}{2}ac \sin B$$
$$\text{Area} = \tfrac{1}{2}bc \sin A$$
3. Where possible you should still use Area $= \frac{1}{2}bh$.
4. Begin worded problems with a diagram and finish them with a written answer.

EXERCISE

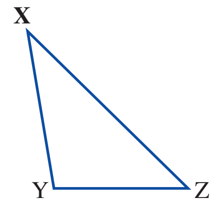
3E Area of a triangle

1 Write down the formula for the area of a triangle in terms of each of the triangles drawn below. Write the formula using the boldfaced angle.

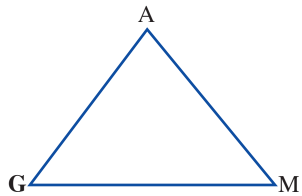
a



b

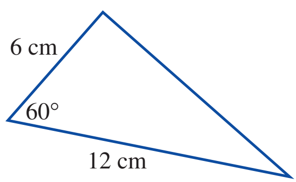


c

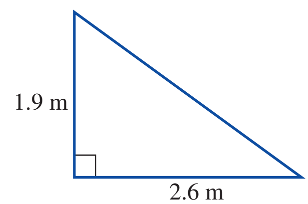


2 For each of the triangles drawn below, state whether the area would be best found using the formula Area $= \frac{1}{2}ab \sin C$ or Area $= \frac{1}{2}bh$.

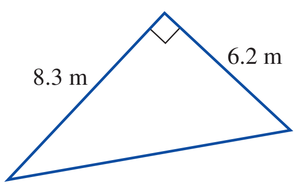
a



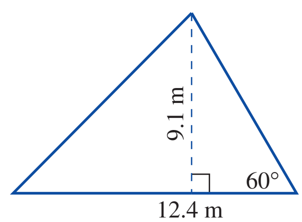
b



c

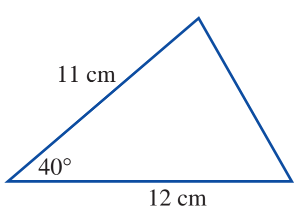


d

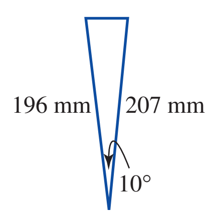


3 WE14 Find the area of each of the following triangles, correct to 1 decimal place.

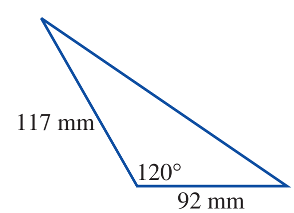
a



b

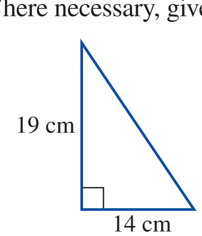


c

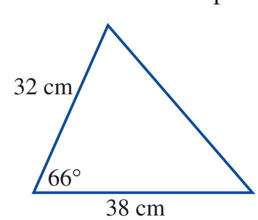


4 Use either Area $= \frac{1}{2}ab \sin C$ or Area $= \frac{1}{2}bh$ to find the area of each of the following triangles. Where necessary, give your answer correct to 1 decimal place.

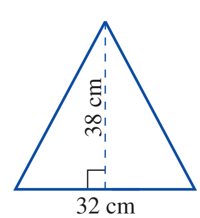
a



b

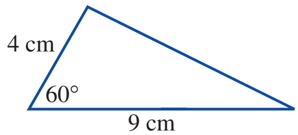


c

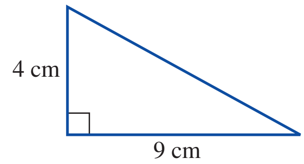


5 **MC** In which of the following triangles can the formula Area $= \frac{1}{2}ab \sin C$ not be used to find the area of the triangle?

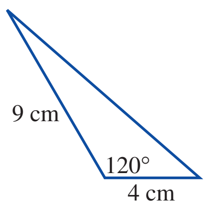
A



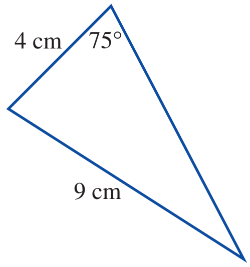
B



C

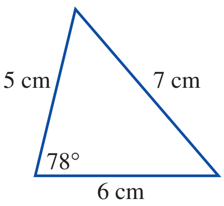


D



6 **MC** The area of the triangle on the right (correct to 1 decimal place) is:

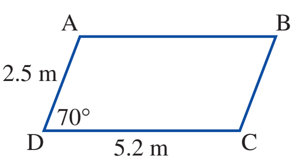
A 4.4 cm^2
B 14.7 cm^2
C 17.1 cm^2
D 20.5 cm^2



7 In ΔPQR, $p = 4.3$ cm, $q = 1.8$ cm and $\angle PRQ = 87°$. Calculate the area of ΔPQR, correct to 4 significant figures.

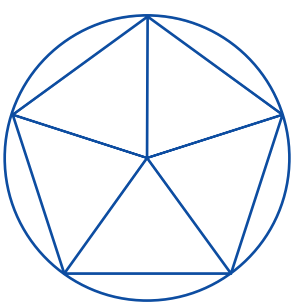
8 The figure on the right is of a parallelogram, ABCD.

a Copy the diagram into your workbook and draw the diagonal AC on your diagram.
b By considering the parallelogram as two equal triangles, calculate its area, correct to 1 decimal place.

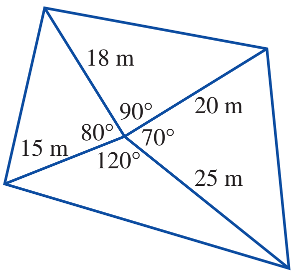


9 On the right is a diagram of a pentagon inscribed in a circle of radius 5 cm.

a Calculate the size of each of the angles made at the centre.
b Calculate the area of the pentagon, correct to the nearest square centimetre.



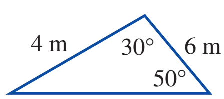
10 **WE15** A surveyor sights the four corners of a block of land and makes the following notebook entry. Calculate the area of the block of land, correct to the nearest square metre.



Further development

11 Find the area of an equilateral triangle with a side length of 10 cm.

12



The correct expression for the area of the shape above is:

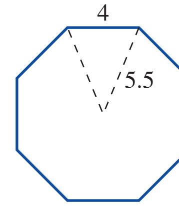
A $\frac{1}{2} \times 6 \times 4 \times \sin 80°$

B $\frac{1}{2} \times 6 \times 4 \times \cos 80°$

C $\frac{1}{2} \times 6 \times 4 \times \sin 30°$

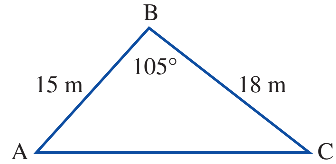
D $\frac{1}{2} \times 6 \times 4 \times \sin 100°$

13 The correct expression for the area of the octagon shown is:



A $11 \sin 45°$
B $88 \sin 67.5°$
C $88 \sin 45°$
D $11 \sin 67.5°$

14 Consider the triangle ABC drawn below.



a Find the area of the triangle.
b Use your answer to part (a) together with the formula $A = \frac{1}{2}\, bh$ to find the shortest distance of the point A from the line BC.

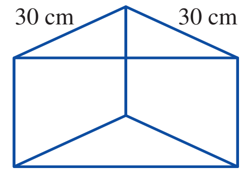
15 The triangle PQR has side lengths PQ = 15 cm, QR = 22 cm and $\angle PQR = 75°$.

a Find the area of the triangle.

b Betty draws the triangle by mistake with $\angle PQR = 105°$. Show that Betty will still get the correct answer.

c Explain why the same answer for the area of the triangle is obtained.

16 Penny is making a triangular display case as shown in the diagram. The two sides are to be 30 cm in length. Find the angle between the two sides that will maximise the area of the triangular cross-section.



3F Using the cosine rule to find side lengths

Finding side lengths

When given the length of one side and two angles in a triangle, we can use the sine rule to find another side length. However, in many cases we do not have this information and need another method of calculating the side lengths. The **cosine rule** allows us to calculate the length of the third side of a triangle when we are given the length of the other two sides and the included angle.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

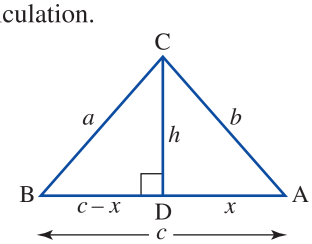
$$c^2 = a^2 + b^2 - 2ab \cos C$$

The formula sheet gives the third version of this formula only. The others are an adaptation of the same rule.

It is important to notice that the formula is given in terms of a^2, b^2 or c^2. This means that to find the value of a, b or c we need to take the square root of our calculation.

Derivation of the cosine rule

Consider ΔABC on the right. In this triangle, h is the perpendicular height of the triangle and meets AB at D. We will let AD = x, and therefore BD = $c - x$.



Using Pythagoras' theorem on ΔBCD: $a^2 = (c - x)^2 + h^2$

$a^2 = c^2 - 2cx + x^2 + h^2$ [1]

From ΔACD: $b^2 = x^2 + h^2$

Therefore: $h^2 = b^2 - x^2$

Substituting for h^2 in [1]: $a^2 = c^2 - 2cx + x^2 + b^2 - x^2$

$a^2 = c^2 - 2cx + b^2$ [2]

Now in ΔACD: $\cos A = \dfrac{x}{b}$

Therefore: $x = b \cos A$

Substituting for x in [2]: $a^2 = c^2 - 2c(b \cos A) + b^2$

$a^2 = c^2 + b^2 - 2bc \cos A$

This becomes the formula for the cosine rule. A similar formula can be used for finding sides b and c. You may like to try it for yourself.

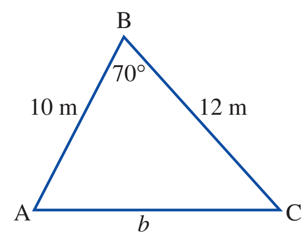
1 Start with ΔABC and draw a perpendicular line from A to BC.

2 Use this diagram and follow the method shown to obtain the following version of the cosine rule: $b^2 = a^2 + c^2 - 2ac \cos B$.

3 Can you obtain $c^2 = a^2 + b^2 - 2ab \cos C$?

WORKED EXAMPLE 16

Find the length of the side marked b in the triangle on the right, correct to 1 decimal place.



THINK | **WRITE**

Method 1: Technology-free

1 Write the formula with b^2 as the subject.

$b^2 = a^2 + c^2 - 2ac \cos B$

2 Substitute $a = 12$, $c = 10$ and $B = 70°$.

$= 12^2 + 10^2 - 2 \times 12 \times 10 \times \cos 70°$

3 Calculate the value of b^2.

$= 161.915$

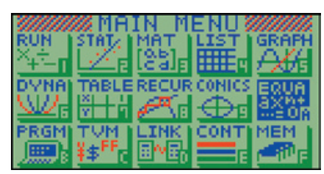
4 Find b by taking the square root of b^2.

$b = \sqrt{161.915}$

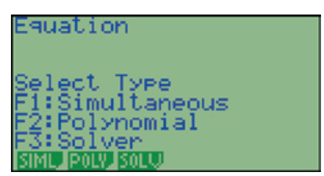
$= 12.7\,\text{m}$

Method 2: Technology-enabled

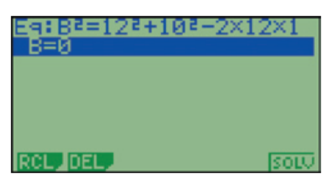
1 From the **MENU** select **EQUA**.



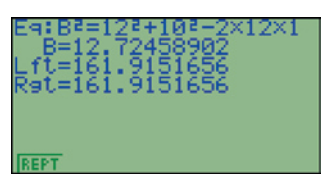
2 Press F3 (**SOLV**).



3 Delete any existing equation, enter **$B^2 = 12^2 + 10^2 - 2 \times 12 \times 10 \times$ cos 70** and then press EXE.



4 Press F6 (**SOLV**) to solve the equation.



As with sine rule questions, we can apply the cosine rule to obtuse-angled triangles. You should recall from the earlier investigation that the cosine ratio of an obtuse angle is negative. The method of solution remains unchanged.

WORKED EXAMPLE 17

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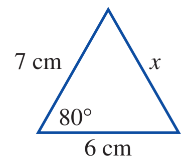
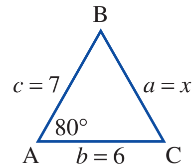
Tutorial
int-0468
Worked example 17

Find the unknown length (to 2 decimal places), x, in the triangle at right.

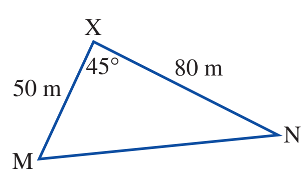


THINK	WRITE
1 Identify the triangle as non-right-angled.	
2 Label the triangle appropriately for the sine rule or cosine rule.	
3 Identify that it is the cosine rule that is required as you have the two sides and the angle in between.	$b = 6 \quad A = 80°$ $c = 7 \quad a = x$
4 Substitute the known values into the cosine rule formula.	$a^2 = b^2 + c^2 - 2bc \times \cos(A)$ $x^2 = 6^2 + 7^2 - 2 \times 6 \times 7 \times \cos(80°)$ $x^2 = 36 + 49 - 84 \times \cos(80°)$ $x^2 = 70.4136$
5 Remember to get the square root value, x.	$x = \sqrt{70.4136}$ $= 8.391$
6 Evaluate the length and include units with the answer.	$x = 8.39$ The unknown length is 8.39 cm, correct to 2 decimal places.

The cosine rule also allows us to solve a wider range of practical problems. The important part of solving such problems is marking the correct information on your diagram. If you can identify two side lengths and the included angle, you can use the cosine rule.

WORKED EXAMPLE 18

A surveyor standing at a point, X, sights a point, M, 50 m away and a point, N, 80 m away. If the angle between the lines XM and XN is 45°, calculate the distance between the points M and N, correct to 1 decimal place.

THINK	WRITE
1 Draw a diagram and mark all given information on it.	

2	Write the formula with x^2 as the subject.	$x^2 = m^2 + n^2 - 2mn\cos X$
3	Substitute $m = 80$, $n = 50$ and $X = 45°$.	$= 80^2 + 50^2 - 2 \times 80 \times 50 \times \cos 45°$
4	Calculate the value of x^2.	$= 3243.15$
5	Calculate x by taking the square root of x^2.	$x = \sqrt{3243.15}$
6	Give a written answer.	$= 56.9$ m

REMEMBER

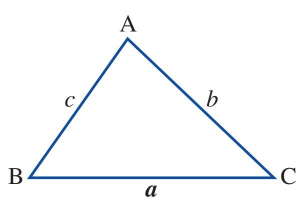
1. To use the cosine rule to find a side length, you need to be given the length of two sides and the included angle.
2. The cosine rule formulas are:
   - $a^2 = b^2 + c^2 - 2bc\cos A$
   - $b^2 = a^2 + c^2 - 2ac\cos B$
   - $c^2 = a^2 + b^2 - 2ab\cos C$.
3. In the solution to cosine rule questions, your final answer is found by taking the square root of the calculation.
4. Begin worded questions by drawing a diagram and finish them by giving a written answer.

EXERCISE

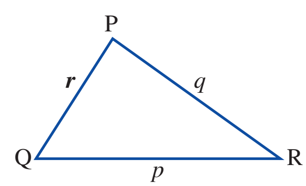
3F Using the cosine rule to find side lengths

1 Write down the cosine rule formula as it applies to each of the triangles below. In each case, make the boldfaced pronumeral the subject.

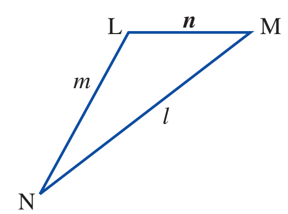
a



b

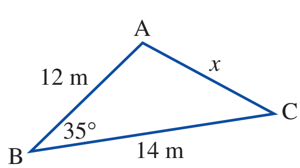


c

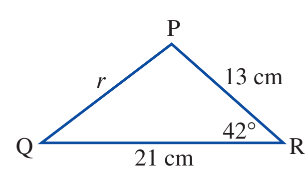


2 WE16 Find the length of the side marked with a pronumeral in each of the following, correct to 3 significant figures.

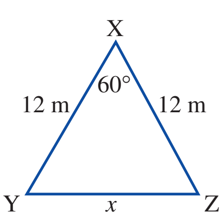
a



b

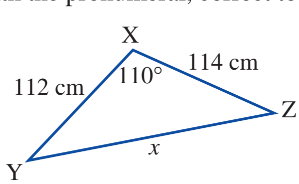


c

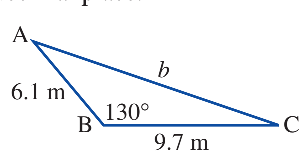


3 WE17 In each of the following obtuse-angled triangles, find the length of the side marked with the pronumeral, correct to 1 decimal place.

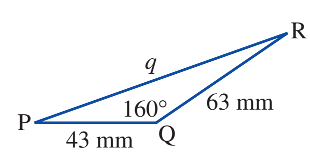
a



b

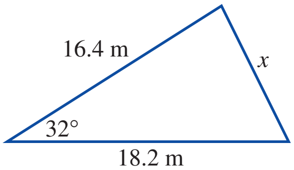


c

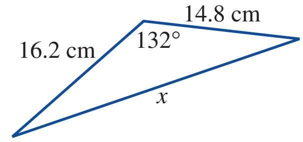


4 MC In which of the following triangles are we unable to use the cosine rule to find x?

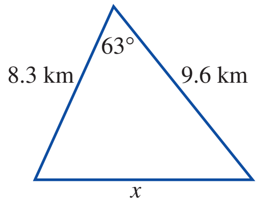
A



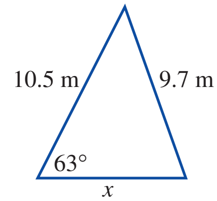
B



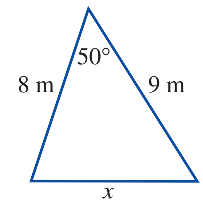
C



D

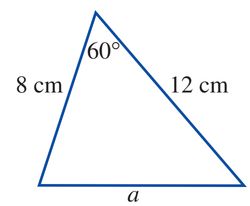


5 MC Look at the triangle drawn on the right. The value of x, correct to 1 decimal place, is:



A 7.2 m **B** 7.3 m
C 52.4 m **D** 52.5 m

6 MC Lieng is asked to find the value of a, correct to 1 decimal place, in the figure drawn on the right. Below is Lieng's solution.



Line 1: $a^2 = 12^2 + 8^2 - 2 \times 12 \times 8 \times \cos 60°$
Line 2: $= 144 + 64 - 192 \times \cos 60°$
Line 3: $= 208 - 192 \cos 60°$
Line 4: $= 16 \times \cos 60°$
Line 5: $= 8$
Line 6: $a = 2.8$ m

Lieng's solution is incorrect. In which line did she make her error?

A Line 2 **B** Line 3 **C** Line 4 **D** Line 5

7 In ΔABC, $a = 14$ cm, $c = 25$ cm and $\angle$ABC $= 29°$. Calculate b, correct to 1 decimal place.

8 In ΔPQR, PQ = 234 mm, QR = 981 mm and $\angle$PQR = 128°. Find the length of side PR, correct to 3 significant figures.

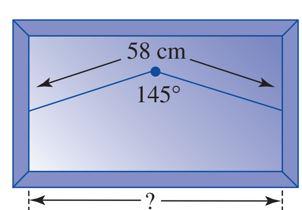
9 WE18 Len and Morag walk separate paths that diverge from one another at an angle of 48°. After three hours Len has walked 7.9 km and Morag 8.6 km. Find the distance between the two walkers at this time, correct to the nearest metre.

10 A cricketer is fielding 20 m from the batsman and at an angle of 35° to the pitch. The batsman hits a ball 55 m and straight behind the bowler. How far must the fieldsman run to field the ball? (Give your answer to the nearest metre.)

11 The sides of a parallelogram are 5.3 cm and 11.3 cm. The sides meet at angles of 134° and 46°.

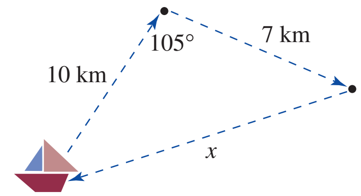
a Draw a diagram of the parallelogram showing this information and mark both diagonals on it.
b Calculate the length of the shorter diagonal, correct to 1 decimal place.
c Calculate the length of the long diagonal, correct to 1 decimal place.

12 The cord supporting a picture frame is 58 cm long. It is hung over a single hook in the centre of the cord and the cord then makes an angle of 145° as shown in the figure on the right. Calculate the length of the backing of the picture frame, to the nearest centimetre.

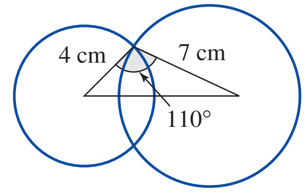


Further development

13 During a sailing race, the boats followed a course as shown. Find the length, x, of its third leg (to 1 decimal place).



14 Two circles, with radii 4 cm and 7 cm, overlap slightly as shown. If the angle between the two radii that meet at the point of intersection of the circumferences is 110°, find the distance between the centres of the circles (to 1 decimal place).



15 Two hikers set out from the same point. One walks N30°E for 1500 metres and the other walks S40°E for 1200 metres. How far apart to the nearest metre are the two hikers?

16 An advertising balloon is attached to two ropes 120 m and 100 m long. The shorter rope makes a 70° angle with the ground and is attached to the bottom of the balloon. The longer rope makes an 80° angle with the horizontal and is attached to the top of the balloon. How tall is the balloon? Give your answer correct to 2 decimal places.

17 A plane takes off at 10.00 am from an airfield and flies at 120 km/h on a bearing of 325°. A second plane takes off from the same airfield and flies on a bearing of 100° at a speed of 90 km/h. How far apart are the planes at 10.25 am?

3G Using the cosine rule to find angles

Finding angles

We can use the cosine rule to find the size of the angles within a triangle. Consider the cosine rule formula.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

We now make $\cos A$ the subject of this formula.

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ a^2 + 2bc \cos A &= b^2 + c^2 \\ 2bc \cos A &= b^2 + c^2 - a^2 \\ \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \end{aligned}$$

In this form, we can use the cosine rule to find the size of an angle if we are given all three side lengths. We should be able to write the cosine rule in three forms depending upon which angle we wish to find.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

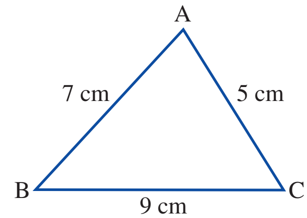
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Again, the formula sheet gives the third version of this formula only. The others are an adaptation of the same rule.

WORKED EXAMPLE 19

Find the size of angle *B* in the triangle on the right, correct to the nearest degree.



THINK | **WRITE**

Method 1: Technology-free

1. Write the formula with cos B as the subject.

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

2. Substitute $a = 9$, $b = 5$ and $c = 7$.

$$\cos B = \frac{9^2 + 7^2 - 5^2}{2 \times 9 \times 7}$$

3. Calculate the value of cos B.

$$\cos B = \frac{105}{126}$$
$$= 0.8333$$

4. Make B the subject of the equation.

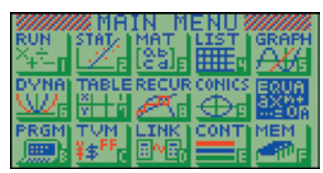
$$B = \cos^{-1}(0.8333)$$

5. Calculate B.

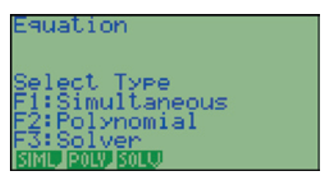
$$B = 34°$$

Method 2: Technology-enabled

1. From the **MENU** select **EQUA**.

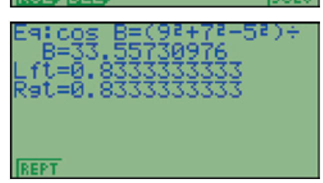


2. Press F3 (**SOLV**).



3. Delete any existing equation, enter the equation **cos B = (9^2 + 7^2 – 5^2) ÷ (2 × 9 × 7)**, and then press EXE.

4. Press F6 (**SOLV**) to solve the equation.



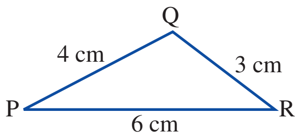
Your formula sheet will give you two versions of the cosine rule, one for finding a side length and one for finding an angle. When using the equation solver it does not matter which version you use to find a side or an angle.

Try using the solver on the equation $\mathbf{5^2 = 9^2 + 7^2 - 2 \times 9 \times 7 \times \cos B}$.

As we found earlier, the cosine ratio for an obtuse angle will be negative. So, when we get a negative result to the calculation for the cosine ratio, this means that the angle we are finding is obtuse. Your calculator will give the obtuse angle when we take the inverse.

WORKED EXAMPLE 20

Find the size of angle Q in the triangle on the right, correct to the nearest degree.



THINK

1. Write the formula with cos Q as the subject.
2. Substitute $p = 3$, $q = 6$ and $r = 4$.
3. Calculate the value of cos Q.
4. Make Q the subject of the equation.
5. Calculate Q.

WRITE

$$\cos Q = \frac{p^2 + r^2 - q^2}{2pr}$$

$$\cos Q = \frac{3^2 + 4^2 - 6^2}{2 \times 4 \times 3}$$

$$\cos Q = \frac{-11}{24}$$
$$= -0.4583$$

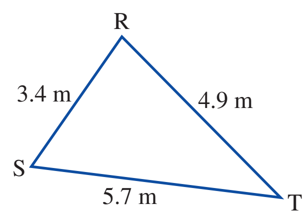
$$Q = \cos^{-1}(-0.4583)$$

$$Q = 117°$$

In some cosine rule questions, you need to work out which angle you need to find. For example, you could be asked to calculate the size of the largest angle in a triangle. To do this you do not need to calculate all three angles. The largest angle in any triangle will be the one opposite the longest side. Similarly, the smallest angle will lie opposite the shortest side.

WORKED EXAMPLE 21

Find the size of the largest angle in the triangle drawn on the right.



THINK

1. ST is the longest side, therefore angle R is the largest angle.
2. Write the formula with cos R the subject.
3. Substitute $r = 5.7$, $s = 4.9$ and $t = 3.4$.

WRITE

$$\cos R = \frac{s^2 + t^2 - r^2}{2st}$$

$$\cos R = \frac{4.9^2 + 3.4^2 - 5.7^2}{2 \times 4.9 \times 3.4}$$

4	Calculate the value of cos R.	$\cos R = \frac{3.08}{33.32}$ $= 0.0924$
5	Make R the subject of the equation.	$R = \cos^{-1}(0.0924)$
6	Calculate R.	$R = 85°$
7	Give a written answer.	The largest angle in the triangle is 85°.

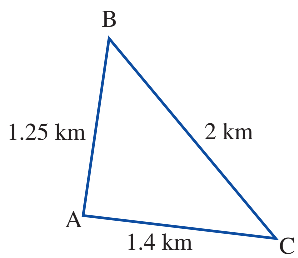
Many problems that require you to find an angle are solved using the cosine rule. As always, these begin with a diagram and are finished off by giving a written answer.

WORKED EXAMPLE 22

Two paths diverge from a point, A. The first path goes for 1.25 km to a point, B. The second path goes for 1.4 km to a point, C. B and C are exactly 2 km apart. Find the angle at which the two paths diverge.

THINK / **WRITE**

1 Draw a diagram.



2 Write the formula with cos A as the subject.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

3 Substitute $a = 2$, $b = 1.4$ and $c = 1.25$.

$$\cos A = \frac{1.4^2 + 1.25^2 - 2^2}{2 \times 1.4 \times 1.25}$$

4 Calculate the value of cos A.

$$\cos A = \frac{-0.4775}{3.5}$$
$$= -0.1364$$

5 Make A the subject of the equation.

$$A = \cos^{-1}(-0.1364)$$

6 Calculate the value of A.

$$= 98°$$

7 Give a written answer.

The roads diverge at an angle of 98°.

REMEMBER

1. The cosine rule formulas are:
   - $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$
   - $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$
   - $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

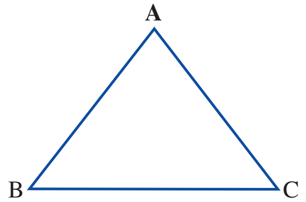
2. If the value of the cosine ratio is negative, the angle is obtuse.
3. In any triangle, the largest angle lies opposite the largest side and the smallest angle lies opposite the smallest side.
4. Worded problems begin with a diagram and end with a written answer.

EXERCISE

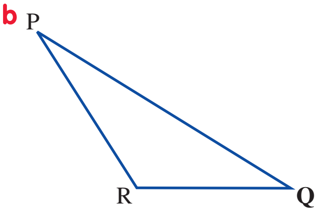
3G Using the cosine rule to find angles

1 For each of the following, write the cosine rule formula as it applies to the triangle drawn with the boldfaced angle as the subject.

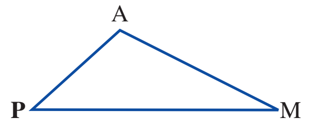
a



b

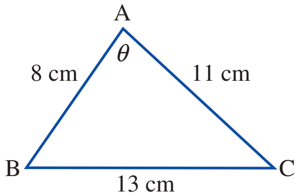


c

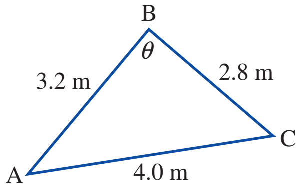


2 WE19 Find the size of the angle marked with the pronumeral in each of the following triangles, correct to the nearest degree.

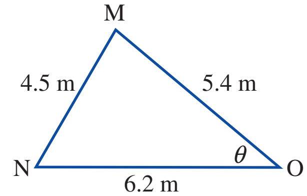
a



b

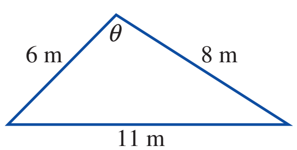


c

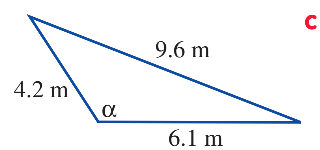


3 WE20 In each of the obtuse-angled triangles below find the size of the angle marked with the pronumeral, to the nearest degree.

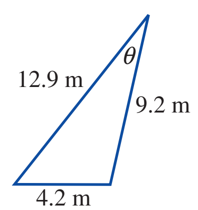
a



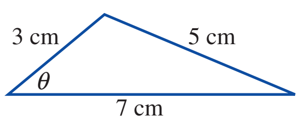
b



c



4 MC Look at the figure drawn below.



Which of the following correctly represents the value of $\cos\theta$?

A $\cos\theta = \dfrac{3^2 + 7^2 - 5^2}{2 \times 3 \times 7}$

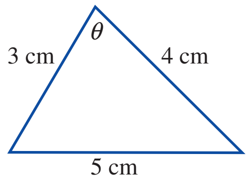
B $\cos\theta = \dfrac{3^2 + 7^2 - 5^2}{2 \times 5 \times 7}$

C $\cos\theta = \dfrac{3^2 + 5^2 - 7^2}{2 \times 3 \times 5}$

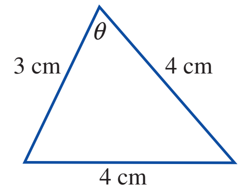
D $\cos\theta = \dfrac{5^2 + 7^2 - 3^2}{2 \times 5 \times 7}$

5 **MC** In which of the following is the angle θ obtuse?

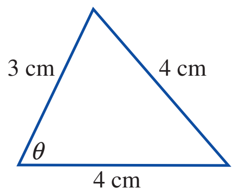
A



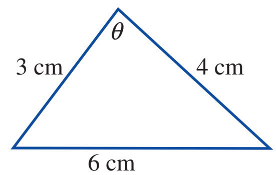
B



C



D



6 In ΔPQR, $p = 7$ m, $q = 9$ m and $r = 6$ m. Find $\angle$QRP, correct to the nearest degree.

7 In ΔKLM, $k = 85$ mm, $l = 145$ mm and $m = 197$ mm. Find the size of the smallest angle, correct to the nearest degree.

8 **WE21** Calculate the size of all three angles (correct to the nearest degree) in a triangle with side lengths 12 cm, 14 cm and 17 cm.

9 WXYZ is a parallelogram. WX = 9.2 cm and XY = 13.6 cm. The diagonal WY = 14 cm.

a Draw a diagram of the parallelogram.

b Calculate the size of $\angle$WXY, correct to the nearest degree.

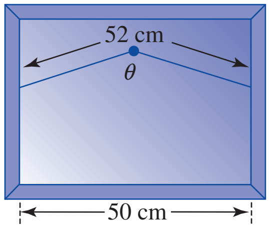
10 **WE22** Two roads diverge from a point, P. The first road is 5 km long and leads to a point, Q. The second road is 8 km long and leads to a point, R. The distance between Q and R is 4.6 km. Calculate the angle at which the two roads diverge.

11 A soccer goal is 8 m wide.

a A player is directly in front of the goal such that he is 12 m from each post. Within what angle must he kick the ball to score a goal?

b A second player takes an angled shot. This player is 12 m from the nearest post and 17 m from the far post. Within what angle must this player kick to score a goal?

12 The backing of a picture frame is 50 cm long and is hung over a picture hook by a cord 52 cm long as shown in the figure on the right. Calculate the angle made by the cord at the picture hook.



Further development

13 Maria cycles 12 km in a direction N68°W and then 7 km in a direction of N34°E.
- **a** How far is she from her starting point?
- **b** What is the bearing of the starting point from her finishing point?

14 A garden bed is in the shape of a triangle, with sides of length 3 m, 4.5 m and 5.2 m.
- **a** Calculate the smallest angle.
- **b** Hence, find the area of the garden.

15 A hockey goal is 3 m wide. When Sophie is 7 m from one post and 5.2 m from the other, she shoots for goal. Within what angle must the shot be made if it is to score a goal?

16 A plane flies in a direction of N70°E for 80 km and then on a bearing of S10°W for 150 km.
- **a** How far is the plane from its starting point?
- **b** What direction is the plane from its starting point?

17 Three circles of radii 5 cm, 6 cm and 8 cm are positioned so that they just touch one another. Their centres form the vertices of a triangle. Find the largest angle in the triangle.

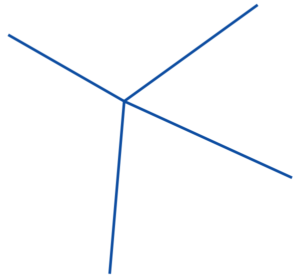
18 From the top of a vertical cliff 68 m high, an observer notices a yacht at sea. The angle of depression to the yacht is 47°. The yacht sails directly away from the cliff, and after 10 min the angle of depression is 15°. How fast is the yacht sailing?

3H Radial surveys

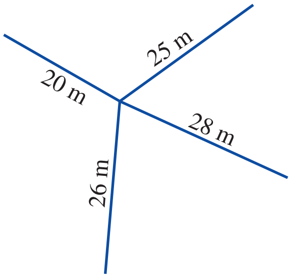
In the preliminary course we examined the **offset survey**. In this survey method an area is measured by drawing a traverse line and measuring offsets at right angles to the traverse line. Because the offset survey created right-angled triangles, the length of each boundary could be calculated using Pythagoras' theorem and the area could be calculated using the formula Area $= \frac{1}{2}bh$.

An alternative survey method to this is a radial survey. One type of radial survey is the **plane table radial survey**. The following steps are taken in a plane table survey.

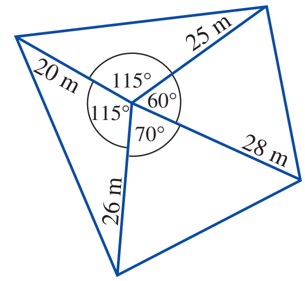
1. A table is placed in the centre of the field to be surveyed, each corner of the field is sighted and a line is ruled on the paper along the line of sight.



2. The distance from the plane table to each corner is then measured.



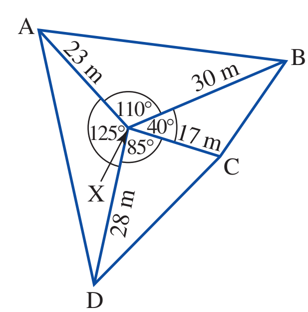
3. The angle between each radial line is then measured and the radial lines joined to complete the diagram.



The field will then be divided into triangles. The length of each side of the field can then be calculated by using the cosine rule. The perimeter of the field is then found by adding the lengths of each side.

WORKED EXAMPLE 23

The figure on the right is a plane table survey of a block of land. Calculate the perimeter of the block of land, correct to the nearest metre.



THINK	WRITE
1 Apply the cosine rule in ΔAXB to calculate the length of AB.	For ΔAXB: $x^2 = a^2 + b^2 - 2ab \cos X$ $= 30^2 + 23^2 - 2 \times 30 \times 23 \times \cos 110°$ $= 1900.99$ $x = 43.6$ m The length of AB is 43.6 m.

2. Apply the cosine rule in ΔBXC to calculate the length of BC.

For ΔBXC:
$$x^2 = b^2 + c^2 - 2bc \cos X$$
$$= 17^2 + 30^2 - 2 \times 17 \times 30 \times \cos 40°$$
$$= 407.63$$
$$x = 20.2 \text{ m}$$
The length of BC is 20.2 m.

3. Apply the cosine rule in ΔCXD to calculate the length of CD.

For ΔCXD:
$$x^2 = c^2 + d^2 - 2cd \cos X$$
$$= 28^2 + 17^2 - 2 \times 28 \times 17 \times \cos 85°$$
$$= 990.03$$
$$x = 31.5 \text{ m}$$
The length of CD is 31.5 m.

4. Apply the cosine rule in ΔDXA to calculate the length of DA.

For ΔDXA:
$$x^2 = d^2 + a^2 - 2da \cos X$$
$$= 23^2 + 28^2 - 2 \times 23 \times 28 \times \cos 125°$$
$$= 2051.77$$
$$x = 45.3 \text{ m}$$
The length of DA is 45.3 m.

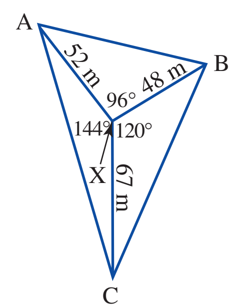
5. Calculate the perimeter by adding the length of each side and rounding the answer to the nearest metre.

$$\text{Perimeter} = 43.6 + 20.2 + 31.5 + 45.3$$
$$= 140.6 \text{ m}$$
$$= 141 \text{ m (correct to the nearest metre)}$$

A similar approach is used to calculate the area of such a field. The area of each triangle is found using the formula Area $= \frac{1}{2}ab \sin C$. The total area is then found by adding the area of each triangle.

WORKED EXAMPLE 24

Calculate the area of the field on the right. Give your answer correct to the nearest square metre.



THINK

1. Calculate the area of ΔAXB.

WRITE

For ΔAXB: Area $= \frac{1}{2}ab \sin X$
$$= \frac{1}{2} \times 48 \times 52 \times \sin 96°$$
$$= 1241.2 \text{ m}^2$$

2. Calculate the area of ΔBXC.

For ΔBXC: Area $= \frac{1}{2} bc \sin X$
$$= \frac{1}{2} \times 67 \times 48 \times \sin 120°$$
$$= 1392.6 \text{ m}^2$$

3. Calculate the area of ΔCXA.

For ΔCXA: Area $= \frac{1}{2} ca \sin X$
$= \frac{1}{2} \times 52 \times 67 \times \sin 144°$
$= 1023.9 \text{ m}^2$

4. Calculate the total area by adding the area of each triangle.

Total area $= 1241.2 + 1392.6 + 1023.9$
$= 3657.7 \text{ m}^2$
$= 3658 \text{ m}^2$ (correct to the nearest m^2)

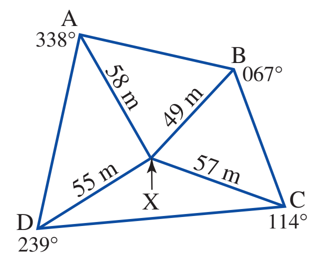
An alternative to the plane table radial survey is the **compass radial survey**. In this survey the bearing of each radial line is calculated and this bearing is used to calculate the angle between each radial, as in the worked example below. The method of calculating the perimeter and area of the field is then the same as for the plane table radial survey.

WORKED EXAMPLE 25

The figure on the right shows a compass radial survey of a block of land.

a Calculate the size of $\angle$AXB.

b Hence, calculate the distance AB, correct to the nearest metre.



THINK

a A is 22° west of North, B is 67° east of North.

b 1. Write the cosine rule formula.
2. Substitute for a, b and X.
3. Calculate the value of x^2.
4. Calculate x.
5. Write your answer.

WRITE

a $22° + 67° = 89°$

b For ΔAXB: $x^2 = a^2 + b^2 - 2ab \cos X$
$= 49^2 + 58^2 - 2 \times 49 \times 58 \times \cos 89°$
$= 5665.8$
$x = 75$ m (correct to the nearest metre)
The distance AB is 75 m.

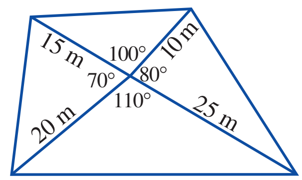
REMEMBER

1. In a radial survey, radial lines are drawn and measured from a point in the centre of an area.
2. In a plane table radial survey, radial lines are drawn on a table by sighting each corner of the field. The length of each line and the angle between the lines is then measured.
3. A compass radial survey is similar but the bearing of each radial line is measured.
4. Each survey divides the area into triangles and the length of each boundary can be calculated using the cosine rule.
5. The area of each triangle can be calculated using the formula Area $= \frac{1}{2} ab \sin C$.

EXERCISE

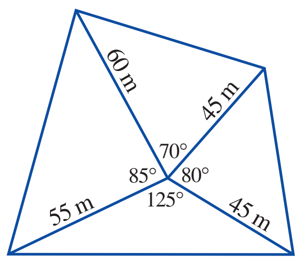
3H Radial surveys

1 WE23 The figure below is a plane table radial survey of a block of land. Use the cosine rule to calculate the perimeter of the block of land, correct to the nearest metre.

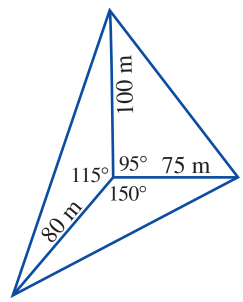


2 Calculate the perimeter of each of the following areas, correct to the nearest metre.

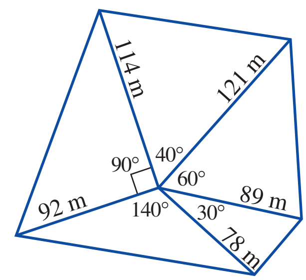
a



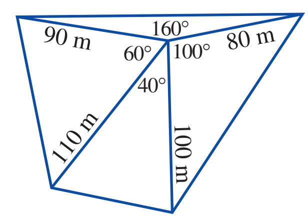
b



c



3 WE24 The figure on the right is a plane table survey of a block of land. Calculate the area of the block, correct to the nearest square metre.

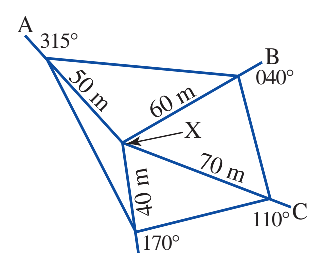


4 For each of the plane table surveys shown in question **2** calculate the area, correct to the nearest square metre.

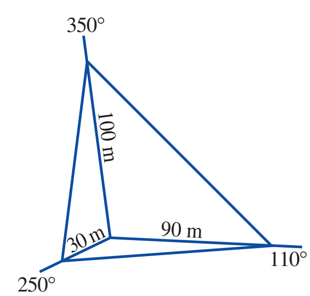
5 WE25 The figure on the right is a compass radial survey of a field.

a Calculate the size of ∠AXB.

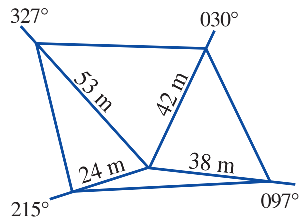
b Hence, use the cosine rule to calculate the distance AB, correct to the nearest metre.

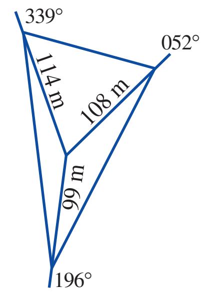


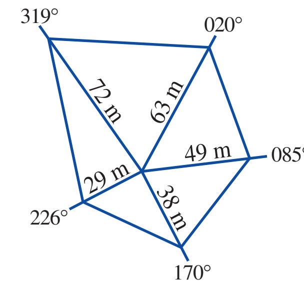
6 Calculate the perimeter of the field given by the compass radial survey on the right. Give your answer correct to the nearest metre.



7 Calculate the perimeter of each of the compass radial surveys shown below.

a


b


c


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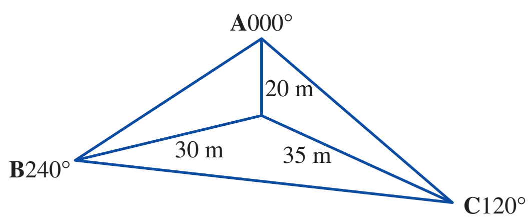
8 For each of the compass radial surveys in question **7** calculate the area, correct to the nearest square metre.

Further development

9 The figure below is that of a triangular field.



a Find the area of the field correct to 2 decimal places.
b Find the perimeter of the field correct to 2 decimal places.
c An alternative formula for the area of a triangle is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b and c are the side lengths of the triangle and $s = \dfrac{a+b+c}{2}$. Use this formula and compare the result with that obtained in question **9a**.

10 ABCD is a square field. O is a point in the centre of the square such that it is 10 metres from each corner.
a Use the cosine rule to find each side length.
b Find the area of triangle OAB and find the area of the square by multiplying this result by the number of triangles that make up the square.
c Compare the answer to part **b** to the area of the square found using $A = l^2$.

11 A surveyor, at point S sights two trees, A and B, on the opposite side of a river as being a distance of 52 m and 64 m away on bearings of 320° and 030° respectively.
a Find the distance AB correct to 2 decimal places.
b Find the area of the triangle ABS correct to 2 decimal places.
c Use your answers to **a** and **b** to find the width of the river.

12 Jason is at point A. Peta is 100 m from Jason on a bearing 090° at point X. Jason observes a tower at point B on a bearing 050° and a tree at a bearing of 150° at Y. Peta notes the bearing of the tower as 310° and tree as 240°.
a Use this information to draw a 1 : 1000 sketch.
b Use this diagram to find the length of:
i AX **ii** BX **iii** AY **iv** YB **v** XY
c An observer is at X. Give the bearing of:
i A **ii** B **iii** Y
d Calculate the area of ABXY to the nearest 100 m^2.

13 Cameron sets up a plane table at O and from this point makes the following notes about the surrounding features A to E.

Feature	Distance to feature	Bearing of feature from O
A	25 m	045°
B	35 m	120°
C	20 m	180°
D	40 m	250°
E	65 m	325°

a What surveying method is Cameron using?
b Draw a neat sketch representing this information. Use a 1 : 1000 scale.
c An observer is at E. From E, what would you expect to be the bearings of A, B, C and D?
d Estimate the area enclosed by the perimeter linking features of A to E. Express to the nearest 10 m^2.

14 Margaret places a plane table directly over point A and sights B, 60 m from her on a bearing of 020°, and D, 50 m away on a bearing of 100°. She moves the plane table to B and notes the bearing of C, 70 m from her, to be 090°.
a Draw a neat 1 : 1000 sketch showing this information.
b Margaret then moves the plan table to C. What bearing is expected of:
i D ii B?
c What is the distance from C to:
i A ii D?
d Determine the area of ABCD to the nearest 100 m^2.

Conducting a radial survey

Choose an appropriate area in or near your school to conduct a radial survey.

1 Set up a table in the centre of the area and tape a large piece of paper to the table.
2 Mark a point in the middle of the piece of paper and sight each corner of the field from this point, ruling a line from the point in that direction.
3 Use a tape or trundle wheel to measure the distance from the table to each corner of the field.
4 Use your protractor to measure the angle between each radial line.
5 Calculate the area and the perimeter of the field.

SUMMARY

Review of right-angled triangles

- The formulas to be used when solving right-angled triangles are:

$$\sin\,\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\,\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

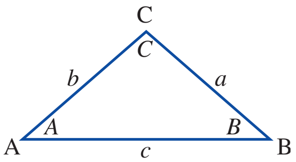
$$\tan\,\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

- To calculate a side length, you need to be given the length of one other side and one angle.
- To calculate the size of an angle, you need to be given two side lengths.
- If a question is given as a problem, begin by drawing a diagram and give a written answer.

Bearings

- Bearings are a measure of direction.
- A compass bearing uses the four main points of the compass, north, south, east and west, as well as the four intermediate directions, north-east, north-west, south-east, south-west.
- More specific directions are given using true bearings. A true bearing describes a direction as a three-digit angle taken in a clockwise direction from north.
- Most bearing questions will require you to draw a diagram to begin the question and require a written answer.

Sine rule



- The sine rule allows us to calculate sides and angles in non-right-angled triangles.
- When finding a side length you need to be given the length of one other side and two angles.
- The sine rule formula is $\dfrac{a}{\sin\,A} = \dfrac{b}{\sin\,B} = \dfrac{c}{\sin\,C}$.
- When finding an angle you need to be given two side lengths and one angle.
- The sine rule formula when finding an angle is $\dfrac{\sin\,A}{a} = \dfrac{\sin\,B}{b} = \dfrac{\sin\,C}{c}$.

Area of a triangle

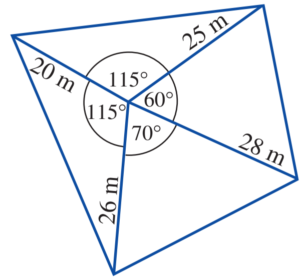
- When you do not know the perpendicular height of a triangle, you can calculate the area using the formula Area $= \frac{1}{2}\,ab\sin C$.
- To calculate the area using this formula, you need to be given the length of two sides and the included angle.

Cosine rule

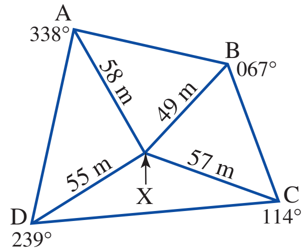
- The cosine rule allows you to calculate the length of sides and size of angles of non-right-angled triangles where you are unable to use the sine rule.
- To find a side length using the cosine rule, you need to be given the length of two sides and the included angle and use the formula $c^2 = a^2 + b^2 - 2ab\cos C$.
- To find an angle using the cosine rule, you need to be given the length of all three sides and use the formula $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

Surveying

- A plane table radial survey sights each corner of a field and draws a radial line in that direction. This divides the field into triangles. The length of each radial line and the angle between radial lines are then measured.



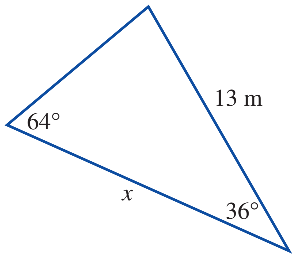
- The cosine rule can then be used to calculate the length of each boundary.
- The formula Area $= \frac{1}{2}ab \sin C$ can be then used to calculate the area of the field.
- A compass radial survey takes the bearing of each radial line and this is then used to calculate the angles between them.



CHAPTER REVIEW

MULTIPLE CHOICE

1 In the figure below, which of the following will give the value of x?



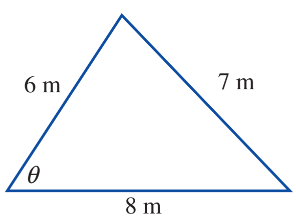
A $x = \dfrac{13 \sin 36°}{\sin 64°}$

B $x = \dfrac{13 \sin 64°}{\sin 36°}$

C $x = \dfrac{13 \sin 64°}{\sin 80°}$

D $x = \dfrac{13 \sin 80°}{\sin 64°}$

2 In the figure below, which of the following will give the value of cos θ?



A $\cos \theta = \dfrac{6^2 + 7^2 - 8^2}{2 \times 6 \times 7}$

B $\cos \theta = \dfrac{6^2 + 8^2 - 7^2}{2 \times 6 \times 8}$

C $\cos \theta = \dfrac{7^2 + 8^2 - 6^2}{2 \times 7 \times 8}$

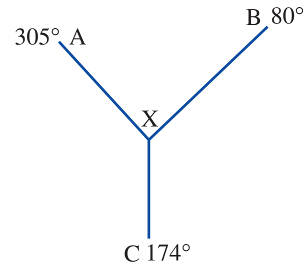
D $\cos \theta = \dfrac{6^2 + 7^2 - 8^2}{2 \times 7 \times 8}$

3 Maurice walks 3 km on a true bearing of 225°. To return to his starting point he must walk on a compass bearing of:

A north-east
B north-west
C south-east
D south-west

4 The figure below is a compass radial survey. $\angle AXB$ is:

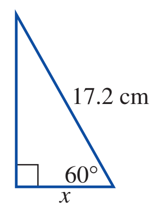
A 35°
B 55°
C 85°
D 135°



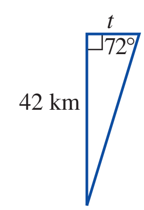
SHORT ANSWER

1 Find the length of the side marked with the pronumeral in each of the right-angled triangles below, correct to 1 decimal place.

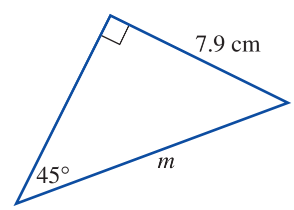
a



b

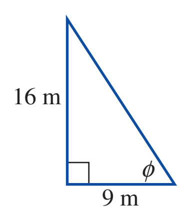


c

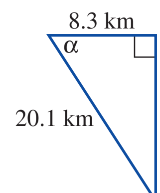


2 In each of the following right-angled triangles, find the size of the angle marked with the pronumeral, correct to the nearest degree.

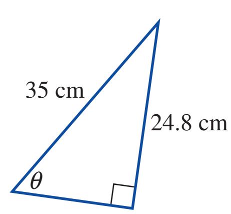
a



b



c



3 An aeroplane at an altitude of 2500 m sights a ship at an angle of depression of 39°. Calculate, to the nearest metre, the horizontal distance from the aeroplane to the ship.

4 When a yacht is 500 m from shore, the top of a cliff is sighted at an angle of elevation of 12°.

a Calculate the height of the cliff, correct to the nearest metre.
b Calculate what the angle of elevation of the top of the cliff will be when the yacht is 200 m from shore.

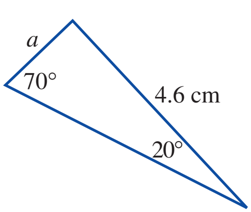
5 Two aircraft are approaching an airport. The Qantas plane (Q) is 40 km due north of the runway (R), while a Jetstar plane (J) is due east of the Qantas plane and north-east of the runway. Calculate the distance of the Jetstar plane from the runway. (Give your answer correct to the nearest metre.)

6 A car rally requires cars to travel for 25 km on a bearing of 240°. The cars are then required to travel due north until they are due west of the starting point. Calculate the distance from the cars to the starting point. (Give your answer correct to 1 decimal place.)

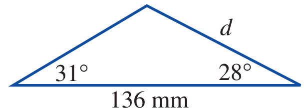
7 A yacht sails due west for 45 nautical miles before turning north for 23 nautical miles.
a Calculate the bearing of the yacht from its starting point.
b On what bearing must the yacht sail to return to its starting point?

8 Use the sine rule to calculate each of the sides marked with a pronumeral, correct to 3 significant figures.

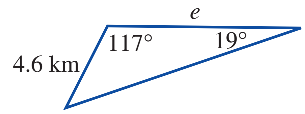
a



b



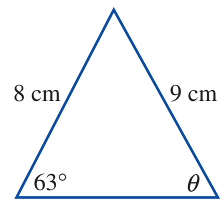
c



9 In ΔXYZ: $x = 9.2$ cm, ∠XYZ = 56° and ∠YXZ = 38°. Find y, correct to 1 decimal place.

10 Use the sine rule to calculate the size of the angle marked with a pronumeral, correct to the nearest degree.

a



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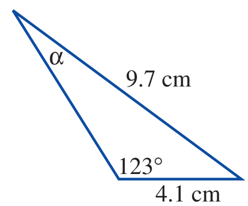

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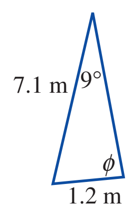
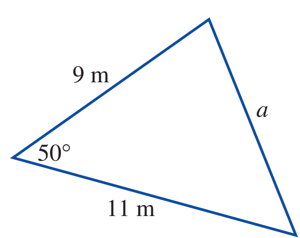


11 In ΔABC: $b = 46$ cm, $c = 37$ cm and ∠BAC = 72°. Find the area of the triangle, correct to the nearest square centimetre.

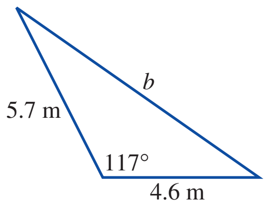
12 Find the area of a triangular field with two sides of 80 m and 98 m, which meet at an angle of 130° (correct to the nearest hundred square metres).

13 Use the cosine rule to find each of the following unknown sides, correct to 3 significant figures.

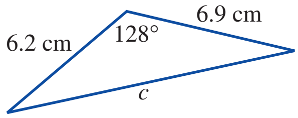
a



b



c

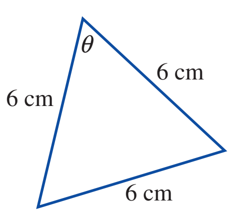


14 In ΔLMN: LM = 63 cm, MN = 84 cm and $\angle$LMN = 68°. Find the length of LN, correct to 1 decimal place.

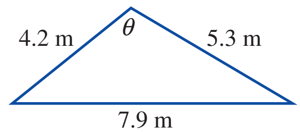
15 During a stunt show two aeroplanes fly side by side until they suddenly diverge at an angle of 160°. After both planes have flown 500 m what is the distance between the planes, correct to the nearest metre?

16 Use the cosine rule to find the size of the angle in each of the following, correct to the nearest degree.

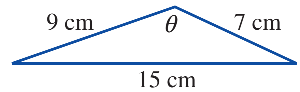
a



b



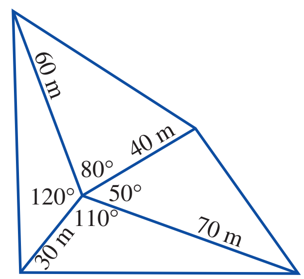
c



17 In ΔXYZ: x = 8.3 m, y = 12.45 m and z = 7.2 m. Find $\angle$YZX, to the nearest degree.

18 Two wooden fences are 50 m and 80 m long respectively. Their ends are connected by a barbed wire fence 44 m long. Find the angle at which the two wooden fences meet.

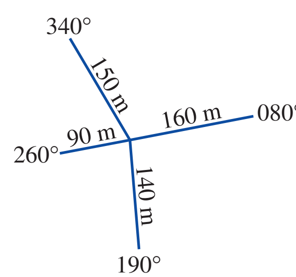
19 The figure below is a plane table radial survey of a field.



a Use the cosine rule to calculate the perimeter of the field.

b Calculate the area of the field.

20 The figure below is a compass radial survey.



a Calculate the perimeter of the field.

b Calculate the area of the field.

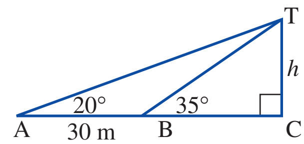
EXTENDED RESPONSE

1 The distance between football goal posts is 7 m. If Soon Ho is 20 m from one goal post and 25 m from the other:

a draw a diagram showing the goal posts and Soon Ho's position.

b calculate the angle within which Soon Ho must kick to score a goal. (Give your answer correct to the nearest degree.)

2 An observer sights the top of a building at an angle of elevation of 20°. From a point 30 m closer to the building, the angle of elevation is 35° as shown in the figure at right.



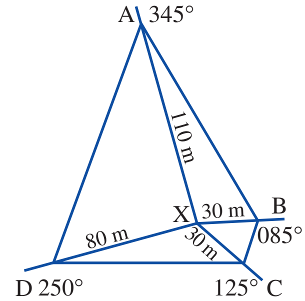
a Calculate the size of ∠ATB.

b Show that the distance BT can be given by the expression $BT = \dfrac{30 \sin 20°}{\sin 15°}$.

c Show that the height of the building can be given by the expression $h = \dfrac{30 \sin 20° \ \sin 35°}{\sin 15°}$.

d Calculate the height of the building correct to 1 decimal place.

3 The figure at right shows a compass radial survey of a field.



a Calculate the length of the boundary CD, correct to 1 decimal place.

b Calculate the area of ΔAXB, correct to the nearest square metre.

eBook*plus* **ACTIVITIES**

Are you ready?

Digital docs *(page 82)*
- SkillSHEET 3.1 (doc-1320): Right-angled trigonometry — finding a side length.
- SkillSHEET 3.2 (doc-1321): Using the inverse trigonometric ratios.
- SkillSHEET 3.4 (doc-1323): Right-angled trigonometry — finding an angle.
- SkillSHEET 3.5 (doc-1324): Converting nautical miles to kilometres.
- SkillSHEET 3.6 (doc-1325): Angle sum of a triangle.
- SkillSHEET 3.7 (doc-1326): Solving fractional equations.

3A Review of right-angled triangles

Tutorial
- WE3 int-2415: Use trigonometric ratios to solve problems. *(page 85)*

Digital docs *(page 87)*
- SkillSHEET 3.1 (doc-1320): Right-angled trigonometry — finding a side length.
- SkillSHEET 3.2 (doc-1321): Using the inverse trigonometric ratios.
- SkillSHEET 3.3 (doc-1322): Rounding angles to the nearest degree.
- SkillSHEET 3.4 (doc-1323) Right-angled trigonometry — finding an angle.

3B Bearings

Interactivity int-0190:
- Navigation and specifications of locations. *(page 89)*

Tutorial
- WE7 int-0473: Apply knowledge of bearings to solve problems. *(page 91)*

Digital doc
- SkillSHEET 3.5 (doc-1324): Converting nautical miles to kilometres. *(page 94)*

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- WE14 int-0469: Calculate the bearing of a ship from a specified position. *(page 107)*

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- WE17 int-0468: Learn how to apply the cosine rule. *(page 114)*

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- WorkSHEET 3.2 (doc-1328): Apply your knowledge of the sine and cosine rules to problems. *(page 128)*

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